



# Université d'Ottawa • University of Ottawa

Faculté des sciences  
Mathématiques et de statistique

Faculty of Science  
Mathematics and Statistics

**MAT 1341A – The Final Exam**

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Please, read the following instructions carefully:

- You have 3 hours to complete this test. You can use the backs of the pages for computations. If you wish, you can detach the last page and use it for computations. In any case (detached or not detached) the last page will not be graded or taken into account while grading.
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THIS SPACE IS RESERVED FOR THE MARKER:

Question	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	Total
Mark																
Out of	1	1	1	1	1	1	1	1	1	1	3	3	4	4	(2)	24

1. F13fs, Q2: Let  $\mathcal{F} = \{f \mid f: \mathbb{R} \rightarrow \mathbb{R}\}$  denote the vector space of all real-valued functions.

Which two of the following are subspaces of  $\mathcal{F}$ ? (1)

$$S = \{f \in \mathcal{F} \mid f(-2)f(2) = 0\}$$

$$T = \{f \in \mathcal{F} \mid f(0) = 0\}$$

$$U = \{f \in \mathcal{F} \mid f(1) = f(0)\}$$

$$V = \{f \in \mathcal{F} \mid f(1) = -1\}$$

mark (X) the correct answer:

A  $S$  and  $T$

B  $S$  and  $U$

C  $S$  and  $V$

D  $T$  and  $U$

E  $T$  and  $V$

F  $U$  and  $V$

D:  $S$  is not a subspace: Take  $f(x) = x+2$  and  $g(x) = x-2$ . Then both  $f, g \in S$ . We have  $(f+g)(x) = 2x$ . So that  $f+g \notin S$ .

$V$  is not a subspace: Take  $f(x) = -x$  and  $g(x) = -1$ . Then both  $f, g \in V$  but  $f+g \notin V$ .

2. F13fs, Q3: Suppose  $\{u, v, w\}$  is a set of vectors in a vector space  $V$ .

Which of the following statements is equivalent to (1)

” $\{u, v, w\}$  is linearly independent”?

I. None of the vectors  $u, v$  or  $w$  is a linear combination of the other vectors in  $\{u, v, w\}$ .

II. None of the vectors  $u, v$  or  $w$  is a multiple of any other single vector in  $\{u, v, w\}$ .

III. If  $a, b, c$  are scalars then  $au + bv + cw = 0$  implies  $a = b = c = 0$ .

IV. If  $a = b = c = 0$ , then  $au + bv + cw = 0$ .

mark (X) the correct answer:

A I. and II.

B I. and III.

C I. and IV.

D II. and III.

E II. and IV.

F III. and IV.

B: The definition of linearly independent set is equivalent to I. or to III.

3. F17fs, Q3: Let  $A$  be a square  $n \times n$  matrix with  $n \geq 2$ .

Which of the following statements are true? (1)

- I. If  $\text{rank}(A) = 1$ , there is just one parameter in the general solution of the system  $Ax = 0$ .
- II. If  $\text{rank}(A) = 1$ , there are  $n - 1$  parameters in the general solution of the system  $Ax = 0$ .
- III. If  $A$  is invertible, the homogeneous system  $Ax = 0$  has a unique solution.
- IV. If the system  $Ax = 0$  has infinitely many solutions, then  $\text{rank}(A) = n$ .

mark (X) the correct answer:

- A I. only
- B II. only
- C I. and III.
- D II. and III.
- E I. and IV.
- F II. and IV.

D: Using the formula  $\text{rank}(A) + \#parameters = \#columns$ , we obtain that if  $\text{rank}(A) = 1$ , then  $\#parameters = n - 1$ , so II. is correct. Also IV. is incorrect by the same formula.

4. F17fs, Q4: Find the polar form of the complex number  $\frac{1-\sqrt{3}i}{i-1}$ . (1)

(see the last page for the table of trigonometric functions)

mark (X) the correct answer:

- A  $\sqrt{2}(\cos(-7\pi/12) + i \sin(-7\pi/12))$
- B  $\sqrt{2}(\cos(11\pi/12) + i \sin(11\pi/12))$
- C  $\sqrt{2}(\cos(5\pi/12) + i \sin(5\pi/12))$
- D  $\sqrt{2}(\cos(\pi/12) + i \sin(\pi/12))$
- E  $\sqrt{2}(\cos(-\pi/12) + i \sin(-\pi/12))$
- F  $\sqrt{2}(\cos(-5\pi/12) + i \sin(-5\pi/12))$

B: We have

$$\frac{1-\sqrt{3}i}{i-1} = \frac{\frac{1}{2} - \frac{\sqrt{3}}{2}i}{-\frac{1}{2} + \frac{1}{2}i} = \frac{e^{i(-\pi/3)}}{\frac{1}{\sqrt{2}}e^{i(3\pi/4)}} = \sqrt{2}e^{i(-\pi/3-3\pi/4)} = \sqrt{2}e^{i(-13\pi/12)} = \sqrt{2}e^{i(11\pi/12)}.$$

5. F11fs, Q7: The dimension of the subspace  $U = \{A \in M_{3 \times 3}(\mathbb{R}) \mid A^T = -A\}$  is (1)

mark (X) the correct answer:

A 0

B 2

C 3

D 4

E 6

F 9

C: We need 3 parameters to describe this subspace as

$$U = \left\{ \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\}.$$

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6. F11fs, Q4: Suppose  $A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{pmatrix}$ . Which one of the following statements is true? (1)

mark (X) the correct answer:

A  $A$  is not invertible.

B The third row of  $A^{-1}$  is  $(-1, -1, 1)$ .

C The second row of  $A^{-1}$  is  $(1, 2, -1)$ .

D The first row of  $A^{-1}$  is  $(2, 0, -1)$ .

E The second column of  $A^{-1}$  is  $(0, 2, -1)^T$ .

F All of B, C, D, E are true.

A: The third row of  $A$  is a sum of the first two. So the rows are linearly dependent, hence,  $A$  is not invertible

7. F16fs, Q7: The set  $\{u, v, w\}$  is an orthogonal set of vectors, where

$$u = (0, 3, 4), \quad v = (1, 0, 0) \quad \text{and} \quad w = (0, 4, -3).$$

If  $(0, -1, -1) = au + bv + cw$ , then  $(a, b, c) =$  (1)

mark (X) the correct answer:

A  $(-\frac{7}{5}, 0, -\frac{1}{5})$

B  $(-\frac{7}{25}, 0, -\frac{1}{25})$

C  $(\frac{7}{25}, 0, \frac{1}{25})$

D  $(-\frac{7}{25}, 0, \frac{1}{25})$

E  $(-7, 0, -1)$

F  $(0, -1, -1)$

B:

$$a_1 = \frac{(0,3,4) \cdot (0,-1,-1)}{3^2+4^2} = -\frac{7}{25}, \quad a_2 = 0, \quad a_3 = \frac{(0,4,-3) \cdot (0,-1,-1)}{3^2+4^2} = -\frac{1}{25}.$$

8. F16fs, Q9: Consider the matrix  $A = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$ . Answer the following questions (Yes/No):

(1)

- Is 3 the only eigenvalue of  $A$ ?
- Is the dimension of the eigenspace corresponding to eigenvalue 3 equal to 1?
- Is  $A$  diagonalizable?

mark (X) the correct answer:

A Yes, Yes, Yes

B Yes, Yes, No

C No, Yes, Yes

D No, Yes, No

E No, No, Yes

F Yes, No, No

B: The eigenvalue is 3 and the dimension is 1.

9. F14fs, Q9: Let  $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$  for some numbers  $a, b, c, d, e, f, g, h, i$ .

If  $\det(A) = 3$ , find the determinant of the matrix  $\begin{pmatrix} 4g & a & d - 2a \\ 4h & b & e - 2b \\ 4i & c & f - 2c \end{pmatrix}$ . (1)

mark (X) the correct answer:

A 6

B -6

C 12

D -12

E 24

F -24

C: perform the row and column operations to reduce it to  $A$ .

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10. F14fs, Q10: Let  $\mathcal{F} = \{f \mid f: \mathbb{R} \rightarrow \mathbb{R}\}$  denote the vector space of real-valued functions.

Which of the following are linearly independent subsets in  $\mathcal{F}$ ? (1)

$$S = \{\sin x, \cos x\}$$

$$T = \{1, \sin x, \cos x\}$$

$$U = \{1, \sin^2 x, \cos^2 x\}$$

$$V = \{1, 2 \sin^2 x, 3 \cos^2 x\}$$

mark (X) the correct answer:

A  $S$  and  $T$

B  $S$  and  $U$

C  $S$  and  $V$

D  $T$  and  $U$

E  $T$  and  $V$

F  $U$  and  $V$

A: We have  $\sin^2 x + \cos^2 x = 1$ .

11. F13fs, Q11: (3 points) Let  $k \in \mathbb{R}$  and consider the linear system in the unknowns  $x, y, z$ :

$$\begin{cases} kx + 2y + z = 0 \\ y + z = 0 \\ 3z = 0 \end{cases}$$

Find all values of  $k$  for which this system has

(a) a unique solution, ANSWER:  (1)

(b) infinitely many solutions, ANSWER:  (1)

(c) no solutions. ANSWER:  (1)

Solution:

As this is a homogeneous system it always has a solution.

So the answer to (c) is  There are no such  $k$ 's.

We compute

$$\det A = \det \begin{pmatrix} k & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{pmatrix} = 3k.$$

The the system has a unique solution if and only if  $A$  is invertible if and only if  $\det A \neq 0$ . So  for (a)  $k \neq 0$ .  For (b)  $k = 0$ .

**12. F14fs, Q12:** (3 points) Let  $v_1 = (1, 0, 0, -1)$ ,  $v_2 = (1, -1, 0, 0)$ ,  $v_3 = (1, 0, 1, 0)$ .

Consider the subspace  $U = \text{Span}\{v_1, v_2, v_3\} \subset \mathbb{R}^4$ .

(a) Explain (you may refer to results learned in class) why  $\{v_1, v_2, v_3\}$  is a basis of  $U$ . (1)

Solution:

It is enough to show that  $\{v_1, v_2, v_3\}$  are linearly independent. We have

$$\text{rank} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix} = \text{rank} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} = 3$$

which is the same as the number of vectors.

(b) Use the Gram-Schmidt algorithm to find an orthogonal basis for  $U$ . (1)

The orthogonal basis is:

write it as row vectors

Solution:

Set  $u_1 = v_1$ ,

$$u_2 = v_2 - \frac{v_2 \cdot u_1}{u_1 \cdot u_1} u_1 = (1, -1, 0, 0) - \frac{1}{2}(1, 0, 0, -1) = \left(\frac{1}{2}, -1, 0, \frac{1}{2}\right).$$

$$u_3 = v_3 - \frac{v_3 \cdot u_1}{u_1 \cdot u_1} u_1 - \frac{v_3 \cdot u_2}{u_2 \cdot u_2} u_2 =$$

$$(1, 0, 1, 0) - \frac{1}{2}(1, 0, 0, -1) - \frac{1/2}{3/2} \left(\frac{1}{2}, -1, 0, \frac{1}{2}\right) =$$

$$(1, 0, 1, 0) - \left(\frac{1}{2}, 0, 0, -\frac{1}{2}\right) - \left(\frac{1}{6}, -\frac{1}{3}, 0, \frac{1}{6}\right) = \left(\frac{1}{3}, \frac{1}{3}, 1, \frac{1}{3}\right)$$

Hence,  $\{(1, 0, 0, -1), (\frac{1}{2}, -1, 0, \frac{1}{2}), (\frac{1}{3}, \frac{1}{3}, 1, \frac{1}{3})\}$  is an orthogonal basis of  $U$ .

(c) Find the best approximation by a vector in  $U$  to the vector  $(0, 1, 1, 1)$ . (1)

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The best approximation is:

write it as a row vector

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Solution:

$$\text{proj}_U(0, 1, 1, 1) =$$

$$\frac{(0,1,1,1) \cdot (1,0,0,-1)}{2}(1, 0, 0, -1) + \frac{(0,1,1,1) \cdot (\frac{1}{2}, -1, 0, \frac{1}{2})}{3/2}(\frac{1}{2}, -1, 0, \frac{1}{2}) + \frac{(0,1,1,1) \cdot (\frac{1}{3}, \frac{1}{3}, 1, \frac{1}{3})}{4/3}(\frac{1}{3}, \frac{1}{3}, 1, \frac{1}{3}) =$$

$$-\frac{1}{2}(1, 0, 0, -1) + \frac{-1/2}{3/2}(\frac{1}{2}, -1, 0, \frac{1}{2}) + \frac{5/3}{4/3}(\frac{1}{3}, \frac{1}{3}, 1, \frac{1}{3}) =$$

$$-\frac{1}{2}(1, 0, 0, -1) - \frac{1}{3}(\frac{1}{2}, -1, 0, \frac{1}{2}) + \frac{5}{4}(\frac{1}{3}, \frac{1}{3}, 1, \frac{1}{3}) = (-\frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \frac{3}{4}) = \frac{1}{4}(-1, 3, 5, 3).$$

13. F15fs, Q13: (4 points) Let  $A = \begin{pmatrix} 1 & 0 & -2 \\ 1 & 3 & 1 \\ 1 & 0 & 4 \end{pmatrix}$ .

(a) Compute and factorize the characteristic polynomial of  $A$  to show that the only eigenvalues of  $A$  are 2 and 3. (1)

The polynomial is:

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{pmatrix} 1 - \lambda & 0 & -2 \\ 1 & 3 - \lambda & 1 \\ 1 & 0 & 4 - \lambda \end{pmatrix} \stackrel{\text{col}_2}{=} (3 - \lambda) \det \begin{pmatrix} 1 - \lambda & -2 \\ 1 & 4 - \lambda \end{pmatrix} = \\ &= (3 - \lambda)((1 - \lambda)(4 - \lambda) + 2) = (3 - \lambda)(4 - 5\lambda + \lambda^2 + 2) = \\ &= (3 - \lambda)(\lambda - 3)(\lambda - 2) = -(\lambda - 3)^2(\lambda - 2). \end{aligned}$$

(b) Find a basis of the eigenspace corresponding to the eigenvalue 2. (1)

The basis is:

write it as row vector(s)

Solution:

$$E_2 = \text{Null}(A - 2I) = \text{Null} \begin{pmatrix} -1 & 0 & -2 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix} = \text{Null} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} = \{(-2s, s, s) \mid s \in \mathbb{R}\}$$

So that  $\{(-2, 1, 1)\}$  is a basis for  $E_2$ .

(c) Find a basis of the eigenspace corresponding to the eigenvalue 3. (1)

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The basis is:

write it as row vector(s)

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Solution:

$$E_3 = \text{Null}(A - 3I) = \text{Null} \begin{pmatrix} -2 & 0 & -2 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} = \text{Null} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \{(-t, s, t) \mid s, t \in \mathbb{R}\}$$

So that  $\{(-1, 0, 1), (0, 1, 0)\}$  is a basis for  $E_3$ .

(d) Find an invertible matrix  $P$  and a diagonal matrix  $D$  such that  $P^{-1}AP = D$ . (1)

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$P =$

$D =$

Solution:

Columns of the matrix  $P$  consist of bases of the respective eigenspaces  $P = \begin{pmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ .

The matrix  $D$  is the diagonal matrix with the respective eigenvalues on the main diagonal

$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

**14. F16fs, Q15:** (4 points) State whether each of the following is (always) true, or is (possibly) false, in the respective box. Provide the respective justification/(counter-)example.

(a) If  $u, v$  and  $w$  are three linearly independent vectors in a vector space  $V$ , then  $u \in \text{Span}\{u+v, v+w\}$

ANSWER (True/False):

(1/2)

Justification/example:

(1/2)

Solution:

False. Suppose  $u = a(u+v) + b(v+w)$  for some  $a, b \in \mathbb{R}$ . Then  $0 = (a-1)u + (a+b)v + bw$ . Since  $\{u, v, w\}$  are linearly independent,  $a-1 = a+b = b = 0$  which is impossible. So  $u$  can not be written as a linear combination of the vectors  $u+v$  and  $v+w$ . Hence,  $u \notin \text{Span}\{u+v, v+w\}$

(b) Let  $A$  be a  $4 \times 3$  matrix with real entries. If the system  $Ax = 0$  has a non-trivial solution, then the set of columns of  $A$  is linearly dependent.

ANSWER (True/False):

(1/2)

Justification/example:

(1/2)

Solution:

True. Let  $u_1, u_2$  and  $u_3$  denote the columns of  $A$ . Let  $v = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$  be the column vector.

Then the condition  $Av = 0$  is equivalent to  $au_1 + bu_2 + cu_3 = 0$ . Since not all  $a, b, c$  are zeros, the column vectors  $u_1, u_2$  and  $u_3$  are linearly dependent.

(c) Let  $A$  be a  $2 \times 3$  matrix. Then every vector in the nullspace  $Null(A)$  of  $A$  is orthogonal to every row of  $A$ .

ANSWER (True/False):  (1/2)

Justification/example: (1/2)

Solution:

True. Write  $A = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$ , where  $r_1$  and  $r_2$  are the rows of  $A$ . Let  $v \in Null(A)$ . Then  $Av = 0$ .  
But

$$Av = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} v = \begin{pmatrix} r_1 \cdot v \\ r_2 \cdot v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

So every vector in  $Null(A)$  is orthogonal to every row of  $A$ .

(d) If  $A$  is an invertible  $3 \times 3$  matrix, then  $A$  is diagonalizable.

ANSWER (True/False):  (1/2)

Justification/example: (1/2)

Solution:

False. Take the matrix  $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$ . Note that 2 is the only eigenvalue of  $A$  and

$$\dim E_2 = \dim Null \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = 2 < 3.$$

So  $A$  is not diagonalizable.

But  $\det A = 8 \neq 0$ , so  $A$  is invertible.

15. F17fs, Q15: (Bonus Question: 2 points) Define a linear transformation  $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  by

$$T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x + y \\ y - z \\ z + x \end{pmatrix}$$

(a) Find the standard matrix of  $T$ .

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The matrix is:

(1)

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Solution:

We have  $T(e_1) = e_1 + e_3$ ,  $T(e_2) = e_1 + e_2$ ,  $T(e_3) = -e_2 + e_3$ .

So after we write each  $T(e_1)$ ,  $T(e_2)$  and  $T(e_3)$  as columns,

we obtain the desired matrix

$$A_T = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix}.$$

(b) Find a basis for the image of  $T$ .

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The basis is:

(1/2)

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Explain/justify why this is a basis:

(1/2)

Solution:

From the lectures we know that  $\text{im}(T) = \text{Span}\{T(e_1), T(e_2), T(e_3)\}$ .

Since  $T(e_1)$ ,  $T(e_2)$  are linearly independent and  $T(e_3) = T(e_1) - T(e_2)$ ,

the set  $\{T(e_1), T(e_2)\}$  is a basis of  $\text{im}(T)$ .

The last page (use it for draft computations only).

Anything written on the last page will not be graded or taken into account while grading.

$$\begin{aligned} \sin\left(\frac{\pi}{6}\right) &= \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}, & \sin\left(\frac{\pi}{3}\right) &= \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \\ \sin\left(\frac{\pi}{4}\right) &= \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}, & \sin(0) &= \cos\left(\frac{\pi}{2}\right) = 0, & \sin\left(\frac{\pi}{2}\right) &= \cos(0) = 1. \end{aligned}$$