Math 208
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Section 2.5 Exponential Function, Part 1: Simple and Compound Interest

1. What is an Exponential function ?

   a) Example: There are 1000 bacteria in a culture at time = 0 hours (start of the measurement) and the number of bacteria double every hour:

Time: t	0hr	1hr	2hr	3hr……..
Number of bacteria: B	1000	2000	4000	8000
		1000×2^1	1000×2^2	1000×2^3

Formula is: $B = 1000\times2^t$ (B = total number of Bacteria, t = elapsed time after start in hours)

This is an Exponential function

   b) Definition of Exponential function: $f(x) = ca^x$ where $a > 0$, $a \neq 1$, c is a constant coef. a is called the Base, x is called the exponent

Ex 1: $y = x^2$ polynomial function – quadratic

Ex 2: $y = 2^x$ is a exponential function

   c) Note: f(x) = Variable$^{\text{Constant}}$ not exponential

   f(x) = Constant$^{\text{Variable}}$ (constant > 0 and constant ≠ 1) exponential

2. Evaluating Exponential forms (by hand and by calculator)

	Calculation by hand	Calculation by calculator
a) Integer Exponents	$(-4)^{2} = (-4)(-4) = 16$	$(-4)^2 = 16$
	$3^{-2} = \frac{1}{3^2} = \frac{1}{9} = 0.111$	$3^{(-2)} = 0.111$
b) Rational exponent (numerator = 1)	$(-32)^{\frac{1}{5}} = \sqrt[5]{-32} = -2$ Because: -2 times itself 5 times = -32	$(-32)^{(1/5)} = -2$
	$(-9)^{\frac{1}{2}}$ DNE Because: no IR times itself 2 times = -9	$(-9)^{1/2}$ Error (DNE)
c) Rational exponent (numerator ≠ 1) Do in 2 steps: Root then power	$(-9)^{\frac{3}{2}}$ $((-9)^{1/2})^3$ DNE	$(-9)^{(3/2)}$ DNE
	$(-27)^{\frac{2}{3}}$ $((-27)^{1/3})^2$	$((-27)\wedge(1/3))\wedge 2 = 9$

$\Rightarrow (-3)^2 = 9$

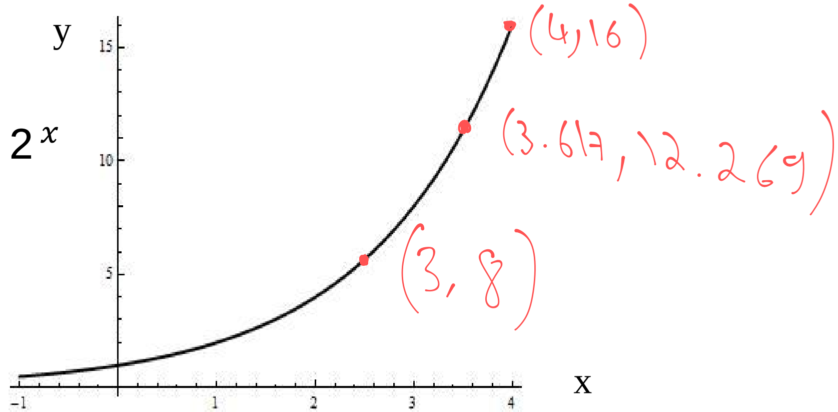
	Calculation by hand	Calculation by calculator
d) Decimal exponent (Rational or Irrational) Base > 0	$2^3 = (2)(2)(2) = 8$ $2^{3.617}$ = ?? $2^4 = (2)(2)(2)(2) = 16$	2 ^ 3 = 8 2^3.617= 12.269 2 ^ 4 = 16

To see what occurs here consider the graph of Exponential function $y = 2^x$, x contained in R

Results above can be interpreted as points on the graph of function $y = 2^x$

x	3	3.617	4
2^x	8	12.269	16

$y = 2^x$



e) Decimal exponent (Rational or Irrational) Base < 0	$(-2)^{\text{even exponent}}$ = +ve $(-2)^{\text{odd exponent}}$ = −ve $(-2)^{\text{decimal exponent}}$ = DNE	even / odd applies to integers only and not decimals. $(-2)^{(3.62)}$ = DNE

- Exponential function $f(x) = a^x$, x contained in R must have $a > 0, a \neq 1$

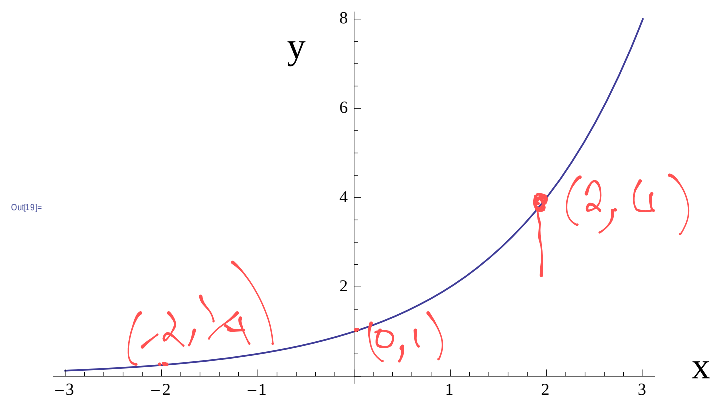
- Calculators use Exponential function for decimal exponents → (−27)^(2/3) = (−27)^(0.66) = DNE

3. Sketching graphs of the Exponential Function

a) The basic exponential function, base a, is $f(x) = a^x$, where $a > 0$, $a \neq 1$

Ex : Consider $y = 2^x$

x	-3	-2	-1	0	1	2	3
f(x)	2^{-3}	2^{-2}	2^{-1}	2^0	2^1	2^2	2^2
	1/8	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8



b) The graph translation ideas for the Quadratic work for all functions (including Exponentials)

ADD c to x ⟶ graph shifts horizontaly left

SUBTRACT c from x ⟶ graph shifts horizontaly right

Replace x with - x ⟶ graph is reflected in y-axis

ADD c to y ⟶ graph shifts vertically DOWN c units

SUBTRACT c from y ⟶ graph shifts vertically UP c units

Replace y with - y ⟶ graph is reflected in x axis

c) If the graph translation patterns do not appear then make a table of values

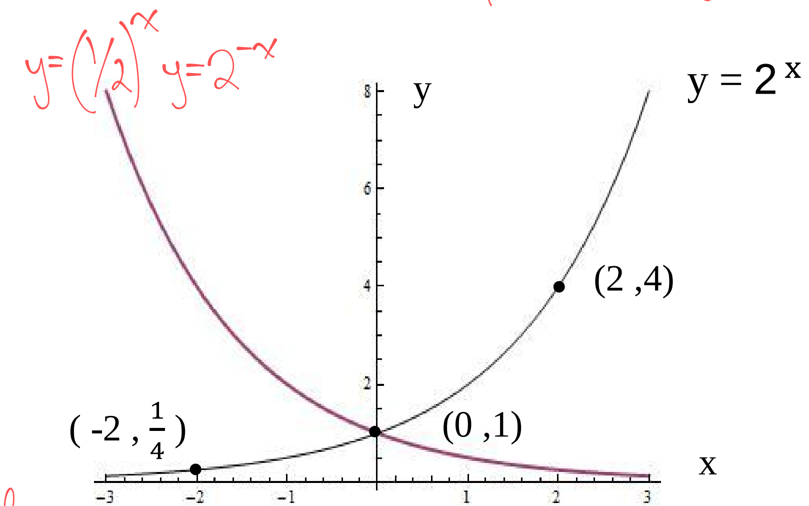
d) Miscellaneous examples

Ex 1 : Sketch graph of $y = (\frac{1}{2})^x$

Note: $y = (\frac{1}{2})^x = (2^{-1})^x = 2^{-x}$

$\rightarrow$ Function $y = (\frac{1}{2})^x$ is same $y = 2^{-x}$ as except x is replaced with $-x$

$\rightarrow$ Graph $y = (\frac{1}{2})^x$ is previous graph $y = 2^x$ reflected in y-axis



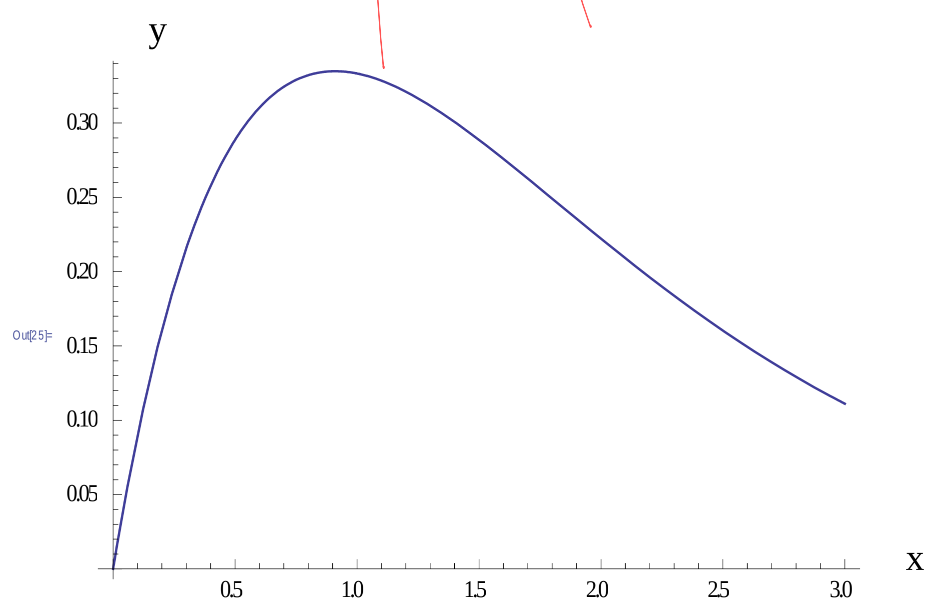
The x axis is the: HA for both graphs

On the graph of $y = (\frac{1}{2})^x$ if $x = -2$, $y = (1/2)^{-2} = \frac{1}{(1/2)^2} = \frac{1}{\frac{1}{4}} = 4$

Ex 2 : Sketch graph of $y = x\,(3^{-x})$ on interval [0, 3]

Because of the multiplication by x the translation ideas do not work.
(make a table of values)

x	0	1	2	3
f(x)	$0(3^{-0})$	$1(3^{-1})$	$2(3^{-2})$	$3(3^{-3})$
	0	$\frac{1}{3} = .33$	.22	.11



I = Interest r = interest rate
P = principle t = time

4. Exponential function and money

a) There are Three types of Interest calculations

Simple interest | Compound Interest | Continious Compound Interest

b) Simple Interest: Formula: I = P r t

Ex: A person makes a purchase of 1000$ and puts the debt on their credit card. The card company charges interest at the rate of 15% per year (Nominal rate) on the unpaid balance. What is the total amount A owed after 5 months ?

$A = P + I \longrightarrow A = P + Prt \longrightarrow A = P(1+rt)$

$A = 1000\left[1 + (.15)\left(\frac{5}{12}\right)\right] \rightarrow A = 1062$

c) Compound Interest:

- If a deposit 1000 $ is made to a bank account where interest is paid at rate of 2 % per year (Nominal rate), what number multiplies the 1000$ after one month ?

Total (after one month) = $1000 + 1000\left(\frac{.02}{12}\right) = 1000\left(1 + \frac{.02}{12}\right) = 1000(1.00166)$

→ amount at start of the month (1000 $) is multiplied by: $\left(1+\frac{.02}{12}\right)$ or (1.00166)

- Compound Interest calculates interest on the total amount present which includes all previous interest earned. ("interest on the interest")

Compound I

start: 1000

1 month: $1000\left(1+\frac{.02}{12}\right)$

2 month: $1000\left(1+\frac{.02}{12}\right)\left(1+\frac{.02}{12}\right.$

Ex1: Consider an investment of 1000\$ that earns 2 % (default is Nominal rate) Compounded monthly. What is the total value A of the investment after 24 months ?

At end of:	Total amount of A
Month 1	$A = \underbrace{1000}_{\text{amt at start}}\left(1+\frac{.02}{12}\right)$
Month 2	$A = \underbrace{[\,1000\,(1+\frac{.02}{12})\,]}_{\text{amt at start month 2}}\left(1+\frac{.02}{12}\right) = 1000\left(1+\frac{.02}{12}\right)^2$
Month 3	$A = \underbrace{[1000\,(1+\frac{.02}{12})^2\,]}_{\text{amt at start month 3}}\left(1+\frac{.02}{12}\right) = 1000\left(1+\frac{.02}{12}\right)^3$
Month 24	$A = 1000\left(1+\frac{.02}{12}\right)^{24}$ ....
	$A =$ .... $\rightarrow 1000\left(1+\frac{.02}{12}\right)^{12(2)}$

Conclusion from form of the final answer:

$A = P\left(1+\frac{r}{m}\right)^{mt}$ m constant

A = Total accumulated amount
P = principle, r = interest rate per year
m = number of compounding periods per year
t = time in years

→ Compound Interest formula

we 360 rather than 365

is an exponential function

Ex 2: Invest 1000$ at 7% compounded daily. How much is in the investment after 5 years ?

$A = P\left(1+\frac{r}{m}\right)^{mt} = 1000\left(1+\frac{.07}{365}\right)^{365(5)} = 1419.02\$$

d) Continuous Compound Interest:

Rewrite the Compound Interest formula in the following form:

$A = P\left[\left(1+\frac{r}{m}\right)^{\frac{m}{r}}\right]^{rt}$

Consider the part inside the square brackets
(use r = .07) and calculate with various m values

Value m	$(1 + \frac{.07}{m})^{\frac{m}{.07}}$
1 yearly	$(1 + \frac{.07}{1})^{\frac{1}{.07}} = 2.6288\ldots$
12 monthly	$(1 + \frac{.07}{12})^{\frac{12}{.07}} = 2.7103\ldots$
365 daily	$(1 + \frac{.07}{365})^{\frac{365}{.07}} = 2.7182\ldots$
8760 hourly	$(1 + \frac{.07}{8760})^{\frac{8760}{.07}} = 2.718\ldots$
∞ (infinity)	

Conclusion: If m → ∞ replace inside square bracket with e (Euler's number)

Then A = Pe^{rt} This is the Continuous Compound Interest formula

Ex : In the previous Ex 2 if the 1000$ at 7% is Compounded Continuously how much is in the investment after 5 years ?

$A = Pe^{rt} = 1000\, e^{.07(5)} \Rightarrow 1419.07$

5. Other uses of Euler's number "e"

a) Any process that causes an amount B to increase (appreciate) or causes an amount B to decrease (depreciate) continuously and not in jumps (spurts) can be expressed using using "e"

For this reason "e" is called the natural base for the exponential function

$B = B_0 e^{rt}$ where B_0 is the initial value of B

b) For depreciation the rate r is taken as a negative number

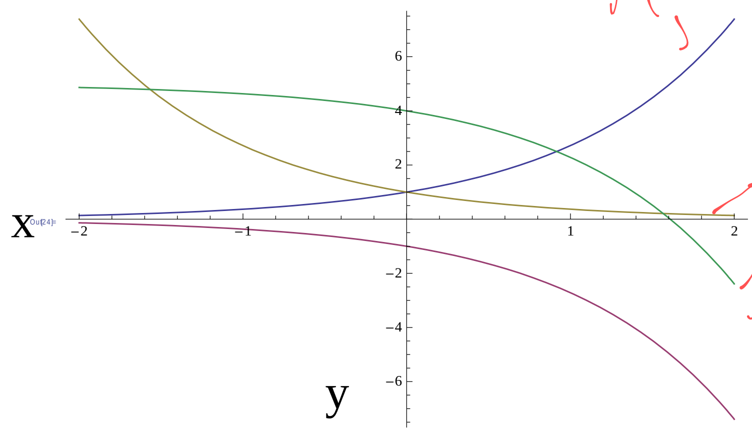
Ex : A car value depreciates continuously at rate of 15% per year and cost 30 000$ at start. What is the estimated value of the car at end of 5 years ?

$V = V_0 e^{rt} = 30{,}000\, e^{-.15(5)} = 14{,}170$

6. Graphs involving "e". Basic Exponential Function, base e, is $f(x) = e^x$

Table of values for Basic Exponential Function:

x	-1	0	1	2
e^x	e^{-1}	e^0	e^1	e^2
≈				



$f(x) = e^{-x}$

$f(x) = -e^x + 5$

$f(x) = -e^x$

$x \rightarrow -x$

$y(x) - 5 = -e^x$

$-y(x) = e^x$

$y \rightarrow -y$

7. Solving equations with exponentials

a) Equation solving is similar to equation solving done to date.

In addition there are several often used ideas to consider:

$e^x = 0$ No solution for x	No x value makes e^x exactly 0
$e^x = 1 \rightarrow x = 0$	
$2^x = -5$ No solution	No x makes 2^x equal a negative number
$(x-1)^5 = (3x-4)^5 \rightarrow x - 1 = 3x - 4$	Exponents equal → bases equal
$(5)^{x^2} = (5)^{-2x+3} \rightarrow x^2 = -2x + 3$	Bases equal → Exponents equal
$(36)^{5x-4} = 6^3 \rightarrow (6^2)^{5x-4} = 6^3$ $10x - 8 = 3$	Sometimes bases can be made equal

b) Miscellaneous examples. Solve for x:

Ex 1 $\quad 10\,x\,e^x - 5\,e^x = 0$

$5e^x(2x - 1) = 0$

$5e^x = 0$	$2x - 1 = 0$
$e^x = 0$	$x = \frac{1}{2}$
No solution	

Factor

Product of 2 factors = 0 → one or both factors = 0

Ex 2 $\quad e^{3x-1} - e = 0$

$e^{3x-1} = e^{1}$

$3x - 1 = 1$

$3x = 2$

$x = \frac{2}{3}$

Bases equal

Ex 3 A person wishes to have 20 000 \$ in an account in 15 years. How much should they invest now if the interest is compounded daily at 4.3 %

$A = P\left(1 + \frac{r}{m}\right)^{mt}$

Compound interest formula

$20{,}000 = P\left(1 + \frac{.043}{365}\right)^{365(15)}$

Solve for P

$P = \frac{20000}{\left(1 + \frac{.043}{365}\right)^{365(15)}} \quad P = 10493.65\$$

Ex 4 Repeat Ex 3 if the interest is compounded continuously

$A = Pe^{rt}$

Continuous Compound interest formula

$20{,}000 = Pe^{.043(15)}$

Solve for P

$P = \frac{20{,}000}{e^{.043(15)}} \Rightarrow 10{,}493.25\$$