LECTURE 6 — Limits of Functions & Continuity Sept 29, 2021

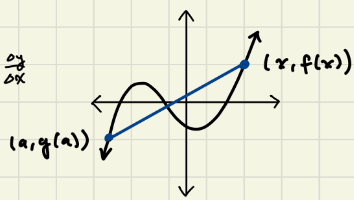
## CHAPTER 4.1 — LIMITS: THE CONCEPT

### Average Rate of Change (AROC)

→ creates secant line between $(a, g(a))$ & $(x, f(x))$

$$\boxed{f(x) = \frac{g(x) - g(a)}{x - a} \quad \text{if } x \neq a}$$

just fancy $\frac{\Delta y}{\Delta x}$



Limits

→ limit of function f as $x$ approaches a is equal to a number L

→ indepencence of f(a)!

↳ doesn't matter what f(a) of if f is defined at a → lim infers what f would like to be at a, not what it is

→ function may not have a limit

$$\boxed{\lim_{x \to a} f(x) = L}$$

RIGHT-HAND LIMIT

$$\lim_{x \to a^+} f(x) = L$$

should equal!

LEFT-HAND LIMIT

$$\lim_{x \to a^-} f(x) = L$$

## CHAPTER 4.3 — ALGEBRIAC LIMIT LAWS

### ALGEBRIAC LIMIT LAWS

I $\lim_{x \to a} c = c$ (if f doesn't depend on $x$, neither does limit)

II $\lim_{x \to a} x = a$

III $\lim_{x \to a} \left(f(x) + g(x)\right) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$

IV $\lim_{x \to a} \left(f(x) g(x)\right) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$

V $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$, if $\lim_{x \to a} g(x) \neq 0$

ex. evaluate $\lim_{x \to 2} (x^3 - 3x + 5)$

$$= \lim_{x \to 2} x^3 - \lim_{x \to 2} 3x + \lim_{x \to 2} 5$$

$$= \left(\lim_{x \to 2} x\right)^3 - 3 \lim_{x \to 2} x + \lim_{x \to 2} 5$$

$$= 2^3 - 3(2) + 5$$

$$= 7$$

## Continuous Function

→ function is continuous at a (in its domain) if $\lim_{x \to a} f(x)$ exists & equals $f(a)$

→ if $\lim_{x \to a} f(x)$ DNE or $\neq f(a)$, $f(x)$ is discontinuous at a

→ few functions are continuous ♡

→ combining continuous functions : if f & g are continuous..

↳ sum, difference, product, & quotient are cont.

↳ $f \circ g$ is cont.

↳ & g is any function where $\lim_{x \to a} g(x) = b$ exists,

$$\lim_{x \to a} (f \circ g)(x) = f\left(\lim_{x \to a} g(x)\right) = f(b)$$

→ if $f(x)$ is continuous, find limit by subbing $a \to f(x)$!

$$\boxed{\lim_{x \to a} f(x) = f(a)}$$ if $f(x)$ is continuous!

## Solving Harder Limits

1. simplify limit
   ↳ rationalize, common denom., cancelling out values
2. once $f(a)$ is defined, sub in a
3. answer will be limit

→ for absolute values, solve limit from both directions

ex. evaluate $\lim_{x \to -3} \frac{|x+3|}{x^2 - 4x - 21}$

FOR LHS LIMIT

$x \to -3^-$, $x < -3$

$x < -3$

$x + 3 < 0$

$\therefore |x+3| = -(x+3)$

$$\lim_{x \to -3^-} \frac{-(x+3)}{(x-7)(x+3)}$$

$$= \lim_{x \to -3^-} -\frac{1}{x-7}$$

$$\boxed{= \frac{1}{10}}$$

FOR RHS LIMIT

$x \to -3^+$, $x > -3$

$x > -3$

$x + 3 > 0$

$|x+3| = x+3$

$$\lim_{x \to -3^+} \frac{x+3}{(x+3)(x-7)}$$

$$= \lim_{x \to -3^+} \frac{1}{x-7}$$

$$\boxed{= -\frac{1}{10}}$$

$$\boxed{\therefore \lim_{x \to -3} f(x) = \text{DNE}}$$

Limits & Vertical Asymptotes

→ if f(x) has vertical asymptote at a, then

$$\boxed{\lim_{x \to a} = \pm \infty}$$

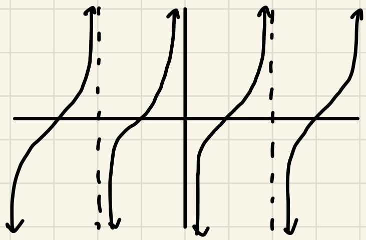
$\lim_{x \to a}$ DNE, but still specify it's infinite

↳ sign of infinity depends on direction of limit

ex. evaluate limit of

$\lim_{x \to \frac{\pi}{2}^-} \tan x = \infty$

$\lim_{x \to \frac{\pi}{2}^+} \tan x = -\infty$




## LECTURE 7 — Continuity, Limits, & ∞ Oct 5, 2021

### CHAPTER 4.8 — LIMITS INVOLVING INFINITY

Infinity: not a #, a concept!

OKAY:

$\infty + \infty = \infty$

$\infty \times \infty = \infty$

$n \times \infty = \infty$, $n > 0$

$n \times \infty = -\infty$, $n < 0$

$\infty \times -\infty = -\infty$

$\frac{1}{\pm\infty} = 0$

MAKES NO SENSE (indeterminant forms):

$\infty - \infty = ??$

$\frac{\infty}{\infty} = ??$

this is illegal! (like $\frac{0}{0}$)

## 4.8.1 – 4.8.2 — Asymptotes

VERTICAL ASYMPTOTES:

→ f(x) gets ∞ly ↑ or ↓ depending on direction of limit

$$\boxed{\lim_{x \to a} f(x) = \infty \quad \text{or} \quad \lim_{x \to a} f(x) = -\infty}$$

↳ NOTE: lim DNE, instead diverges to ∞

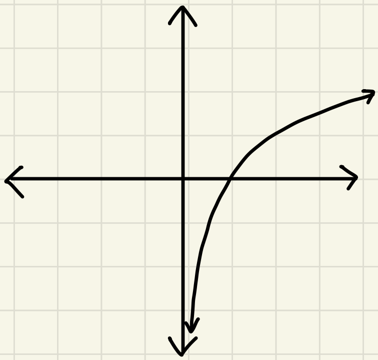
HORIZONTAL ASYMPTOTES:

→ describes longterm behaviour of graph

→ lim of f(x) as $x \to \pm\infty$ ...

$$\boxed{\lim_{x \to \infty} f(x) = L}$$

ex. $\lim_{x \to \infty} \ln x = ?$



$\lim_{x \to \infty} \ln x = \infty$ or:

" as x approaches an infinitely large #, y approaches infinitely"

NOTE: can't use lim laws since they only work for lims that exist

## 4.8.3 — Methods for Finding Lims at ± ∞

METHOD 1: factoring out dominant term

ex. $\lim_{x \to \infty} (3000x^2 - 4x^3)$

$= \lim_{x \to \infty} (-4x^3)(-750\frac{1}{x} + 1)$

approaches $-\infty$

approaches 1

$-\infty \cdot 1 = -\infty$

$= -\infty$

METHOD 2: factor num. & denom. by highest power term in denom.

ex. $\lim_{x \to \infty} = \frac{3x^2+2x+1}{-x^2+3}$ ) factor out $x^2$

$= \frac{\cancel{x^2}(3+2x^{-1}+x^{-2})}{\cancel{x^2}(3x^{-2}-1)}$ ) cancel out $x^2$

$= \frac{3+\cancel{2x^{-1}}^{0}+\cancel{x^{-2}}^{0}}{\cancel{3x^{-2}}^{0}-1}$ ) all other terms $\to 0$

$= -3$

METHOD 3: scaling w/ exponentials

→ divide numerator & denominator by dominant term

ex. $\lim_{x \to \infty} \frac{e^x-1}{4+5e^x}$ ) divide num & denom by $e^x$

$= \lim_{x \to \infty} \frac{1-\cancel{e^{-x}}^{0}}{\cancel{4e^{-x}}^{0}+5}$ ) substitute $\infty$

$= \frac{1}{5}$

METHOD 4: scaling w/ radicals

ex. $\lim_{x \to \infty} \frac{x-2}{\sqrt{3x^2+4}}$ ) sub out leading $(x^2)$ term from term in rad

$= \lim_{x \to \infty} \frac{x-2}{\sqrt{x^2(3+4x^{-2})}}$ ) bring $x^2$ out of rad

$= \lim_{x \to \infty} \frac{x-2}{|x|\sqrt{3+4x^{-2}}}$ ) divide num & denom by $x$

$= \lim_{x \to \infty} \frac{1-\cancel{2x^{-1}}^{0}}{\sqrt{3+\cancel{4x^{-2}}^{0}}}$ ) sub in $\infty$

$= \frac{1}{\sqrt{3}}$

METHOD 5: combining continuos functions (see 4.4 – 4.5)

ex. $\lim_{x\to\infty} e^{-x}$

since $e^{-x}$ is continuous...

$= e^{\left(\lim_{x\to\infty} \frac{1}{x}\right)}$

$\frac{1}{x} \to 0$

$= e^{0}$

simplify

$= 1$

ex. $\lim_{x\to\infty} \left(\ln(x+3) - \ln(x-1)\right)$

ln laws

$= \lim_{x\to\infty} \left(\ln \frac{x+3}{x-1}\right)$

$\ln \frac{x+3}{x-1}$ is continuous

$= \ln\left(\lim_{x\to\infty} \frac{x+3}{x-1}\right)$

div by $x$

$= \ln\left(\lim_{x\to\infty} \frac{1+3\cancel{x^{-1}}^{\,0}}{1-\cancel{x^{-1}}^{\,0}}\right)$

sub $\infty$

$= \ln(1)$

simplify

$= 0$