MAT1308A Lecture note 14, November 6, 2020 Fall term

APPLICATION OF IMPLICIT DIFFERENTIATION

1. REVIEW IMPLICIT DIFFERENTIATION.

- Start equation with variables ($x's$ and $y's$).
- Implicit differentiate both sides of equation. Treat $y's$ as a "mystery" function of x (implicit).

  For the derivative of y, just write $\frac{dy}{dx}$, differentiate $x's$ as usual. Then isolate $\dfrac{dy}{dx}$.

2. APPLICATION OF IMPLICIT DIFFERENTIATION-RELATED RATE:

- We may be interested in rate of change of something or things (with respect time).
- The things (a variables) may be related by some equation.
- If we implicitly differentiate the equation with respect to time, we get a new equation that relates that rates of change.

Example 14.1: A ladder leaning against a wall in 5 m long. The top of the ladder is sliding down at a rate of 2m/s. When the bottom of the ladder is 4 m away from the wall, how fast is the bottom sliding away from the wall.

Solution: To solve these kind questions, please remember x and y are function of time, so they are $x(t), y(t)$. Suppose from the top of the ladder to the ground is y, and the bottom of the ladder to the wall is x. According to Pythagorean theorem, we have

$$x^2 + y^2 = 5^2 \tag{1}$$

Since the top of the ladder is sliding down at a rate of 2m/s, therefore $\dfrac{dy}{dt} = -2\ m/s$. We want to $\dfrac{dx}{dt} =?$ when $x = 4$.
First, when $x = 4$, from (1), we have $y = 3$.

Second, we treat x, y as function of t, $x(t), y(t)$. Using implicit derivative, we take derivative about t for (1) in both sides:

$$\left(x^2(t) + y^2(t)\right)'_t = (5^2)'_t$$

$$2x(t)\frac{dx}{dt} + 2y(t)\frac{dy}{dt} = 0$$

$$\frac{dx}{dt} = -\frac{y}{x}\frac{dy}{dt} \tag{2}$$

Plug known into implicity differentiation equation (2) , we have

$$\frac{dx}{dt} = -\frac{y}{x}\frac{dy}{dt} = -\frac{3}{4}(-2) = 1.5$$

Therefore, the bottom slides out at a rate of $1.5\ m/s$ when the bottom of the ladder is 4 m away from the wall.

Example 14.2: A stone is tossed into a pond creating a circular ripple that grows outward from the center, if the radius of the circle grows at a rate of 10 cm/s.
(a). How fast is the area of the circle growing when the radius is $\frac{100}{\pi}$ cm ?

Solution: Suppose radius is $R(t)$, and area is $A(t)$
Given the radius of the circle grows at a rate of 10 cm/s, which means:
$\frac{dR}{dt} = R'(t) = 10\ cm/s.$
We wan to $\frac{dA(t)}{dt}$ when $R = \frac{100}{\pi}$.
We have relation between variables (Area and radius):

$$A(t) = \pi R^2(t) \tag{3}$$

Both sides derivative about t for (3) , we have

$$A'(t) = 2\pi R(t)\frac{dR}{dt} \tag{4}$$

Plug known into implicit differentiation function (4)

$$A'(t) = 2\pi R(t)\frac{dR}{dt} = 2\pi\frac{100}{\pi}(10) = 2000\ cm^2/second.$$

(b) When the radius is 1 cm?

Plug radius is 1 cm, we have

$$A'(t) = 2\pi R(t)\frac{dR}{dt} = 2\pi(1)(10) = 20\pi\ cm^2/second.$$

Example 14.3: The sides of a rectangle are growing:
Width grows 5 cm/min, length grows 7 cm/min.
How fast is the area growing when width is 10 cm and its length is 14 cm?

Solution: Suppose the rectangle length is $L(t)$, and width is $W(t)$. The area $A(t)$is

$$A(t) = L(t)W(t) \tag{5}$$

Given $\frac{dL}{dt} = L'(t) = 7\ cm/mim,\ \frac{dW}{dt} = W'(t) = 5\ cm/min.$
Both sides take derivative about t for (5), we have

$$A'(t) = L'(t)W(t) + L(t)W'(t) \tag{6}$$

Plug known into implicit differentiation equation (6), we have

$$A'(t) = L'(t)W(t) + L(t)W'(t) = 7(10) + 14(5) = 140\ cm^2/min.$$

The area growing when width is 10 cm and its length is 14 cm is $140\ cm^2/min.$

Elasticity

Assume that we have a demand equation

$$q = f(p)$$

where q stands for the number of items we would sell(per week, per month, or what have you) if we set the price per item at p.

We have following: Price Elasticity of Demand
The price elasticity of demand E is the percentage rate of decrease of demand per percentage increase in price. E is given by the formula

$$E = -\frac{dq/dp}{q/p}$$

We say that the demand is elastic if $E > 1$, is inelastic if $E < 1$, and has unit elasticity if $E = 1$.

Summary: Implicit differentiation; Application of implicit differentiation-Related Rate, Section 5.5.