

Solution to Test 1 (version A)

MAT1320B, Fall 2019

Total = 20 marks

Part I. Multiple-Choice Questions (6 × 2 = 12 marks)**DDAEFA**1. Assume some values of a one-to-one function $y = f(x)$ are given in the following table

x	1	2	3	4	5
$f(x)$	4	5	1	2	3
$g(x)$	2	4	5	3	1

Which one of the following statements is true?

- (A) $(f \circ g)(2) = 1, f^{-1}(2) = 1.$
 (B) $(f \circ g)(2) = 2, f^{-1}(2) = 5.$
 (C) $(f \circ g)(2) = 1, f^{-1}(2) = 4.$
 (D) $(f \circ g)(2) = 2, f^{-1}(2) = 4.$
 (E) $(f \circ g)(2) = 1, f^{-1}(2) = 5.$
 (F) $(f \circ g)(2) = 2, f^{-1}(2) = 1.$

Solution. (D) $(f \circ g)(2) = f(g(2)) = f(4) = 2.$ Since $f(4) = 2, f^{-1}(2) = 4.$ 2. Let $f(x) = \begin{cases} ax + 4, & x \leq 2, \\ ax^2 - 8, & x > 2. \end{cases}$ For which value of a is $f(x)$ continuous for all real numbers?

- (A) 1; (B) 2; (C) 4; (D) 6; (E) 8; (F) 9.

Solution. (D) This function is continuous when $x < 2$ and when $x > 2.$ When $x = 2,$

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} (ax + 4) = 2a + 4, \text{ and } \lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} (ax^2 - 8) = 4a - 8. \text{ Hence, } 2a + 4 = 4a - 8,$$

$$2a = 12, a = 6.$$
3. If $3^x = 2^{x+1},$ then $x =$

- (A) $\frac{\ln 2}{\ln(3/2)};$ (B) $\frac{\ln(3/2)}{\ln 2};$ (C) $\frac{\ln 3}{\ln(3/2)};$
 (D) $\frac{\ln(3/2)}{\ln 3};$ (E) $\ln \frac{4}{3};$ (F) $\ln \frac{3}{2}.$

Solution. (A) Take the natural logarithm on both sides of the equation: $\ln 3^x = \ln 2^{x+1}$.

By the property of the logarithm, $x \ln 3 = (x + 1) \ln 2$. $x(\ln 3 - \ln 2) = \ln 2$.

$$x = \frac{\ln 2}{\ln 3 - \ln 2} = \frac{\ln 2}{\ln(3/2)}.$$

4. Find the one-side limit $\lim_{x \rightarrow -1^-} \frac{2x^2 - x - 3}{|x + 1|}$.

(A) 1; (B) -1; (C) 3; (D) -3; (E) 5; (F) -5.

Solution. (E) $\lim_{x \rightarrow -1^-} \frac{2x^2 - x - 3}{|x + 1|} = \lim_{x \rightarrow -1^-} \frac{(2x - 3)(x + 1)}{-(1 + x)} = -\lim_{x \rightarrow -1^-} (2x - 3) = 5$.

5. If $f(x) = e^{-2x} \sin\left(\frac{\pi}{2}x\right)$, then the equation of the tangent line of the graph of $f(x)$ at the point $(1, e^{-2})$ is

(A) $y = e^{-2}x$; (B) $y = 2e^{-2}x - e^{-2}$; (C) $y = -e^{-2}x + 2e^{-2}$;
 (D) $y = 3e^{-2}x - 2e^{-2}$; (E) $y = -3e^{-2}x + 4e^{-2}$; (F) $y = -2e^{-2}x + 3e^{-2}$.

Solution. By the chain rule, $\frac{d}{dx} e^{-2x} = -2e^{-2x}$, and $\frac{d}{dx} \sin\left(\frac{\pi}{2}x\right) = \frac{\pi}{2} \cos\left(\frac{\pi}{2}x\right)$. Hence, by the product rule, $f'(x) = -2e^{-2x} \sin\left(\frac{\pi}{2}x\right) + \frac{\pi}{2} e^{-2x} \cos\left(\frac{\pi}{2}x\right)$. When $x = 1$, $f'(1) = -2e^{-2}$. The equation of the tangent line of the graph of $f(x)$ at the point $(1, e^{-2})$ is $y = -2e^{-2}(x - 1) + e^{-2}$, or $y = -2e^{-2}x + 3e^{-2}$.

6. Suppose a function $y = f(x)$ is defined implicitly by the equation $\frac{x}{y} + 2x + y^2 = 3$ near a point $(2, -1)$. Then the derivative of this function at the point $(2, -1)$ is

(A) $\frac{1}{4}$; (B) 1; (C) $\frac{2}{3}$; (D) $-\frac{1}{4}$; (E) -2; (F) $\frac{1}{2}$.

Solution. (A) Taking the derivative on both sides with respect to x , we have

$$\frac{y - xy'}{y^2} + 2 + 2yy' = 0. \text{ When } x = 2, \text{ and } y = -1, -1 - 2y' + 2 - 2y' = 0. \text{ } y' = \frac{1}{4}.$$

Part II. Detailed Answer Question (8 marks)

1. (4 marks) Consider function $f(x) = \frac{x^2 - x}{3x^2 - x - 2}$.

Find the horizontal/vertical asymptote(s) of this graph, if any. (Answer NONE if there is no horizontal or vertical asymptote.)

Answer. The horizontal asymptote(s) of the graph of $f(x)$ is / are $y = 1/3$.

The vertical asymptote(s) of the graph of $f(x)$ is/ are $x = -2/3$.

Justification. $\lim_{x \rightarrow \infty} \frac{x^2 - x}{3x^2 - x - 2} = \lim_{x \rightarrow \infty} \frac{1 - 1/x}{3 - 1/x - 2/x^2} = \frac{1}{3}$, and

$\lim_{x \rightarrow -\infty} \frac{x^2 - x}{3x^2 - x - 2} = \lim_{x \rightarrow -\infty} \frac{1 - 1/x}{3 - 1/x - 2/x^2} = \frac{1}{3}$. The graph of $f(x)$ has one horizontal asymptote $y = \frac{1}{3}$.

Let $3x^2 - x - 2 = 0$. $x = 1, x = -\frac{2}{3}$. Since the numerator is zero at $x = 1$, and it is not zero at $x = -\frac{2}{3}$, $x = 1$ is not vertical asymptote and the graph of $f(x)$ has only one vertical asymptotes $x = -\frac{2}{3}$.

2. (4 marks) (a) (1 mark) Give the definition of the derivative of a function $y = f(x)$ at a point $x = a$.

(b) (3 marks) **Use the definition of the derivative** to find the derivative of the function $y = \sqrt{3x - 2}$ at $x = 2$.

Solution. (a) $f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$.

(b) $y'(2) = \lim_{h \rightarrow 0} \frac{1}{h} (\sqrt{3(2+h) - 2} - 2) = \lim_{h \rightarrow 0} \frac{(\sqrt{4+3h} - 2)(\sqrt{4+3h} + 2)}{h(\sqrt{4+3h} + 2)} = \lim_{h \rightarrow 0} \frac{3}{\sqrt{4+3h} + 2} = \frac{3}{4}$.