MAT1320C CALCULUS I

4. Limits, Limits at Infinity, Continuity, and I.V.T. (Sept. 2 0)

Lec 3 mini review.

slope of secant: $\frac{f(b)-f(a)}{b-a}$

average rate of change: $\frac{f(b)-f(a)}{b-a}$

goal: slope of tangent at a:

goal: instantaneous rate of change at a:

$\frac{f(a+h)-f(a)}{h}$ want $h \to 0$

$\frac{f(a+h)-f(a)}{h}$ want $h \to 0$

limits: the intuitive definition

$\lim_{x\to a} f(x) = L$

one-sided limits:

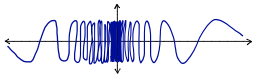
$\lim_{x\to a^-} f(x)$ $\lim_{x\to a^+} f(x)$

why some limits DNE:
infinite limits (vertical asymptotes)
no unique real number L
different or DNE from left/right

ways to evaluate limits:
numerically graphically
with **Limit Laws** and algebraic tricks
(factoring, rationalizing,...)

SQUEEZING LIMITS

Example 4.1. Recall from last class that $\lim_{x\to 0} \sin\left(\frac{\pi}{x}\right)$ DNE.



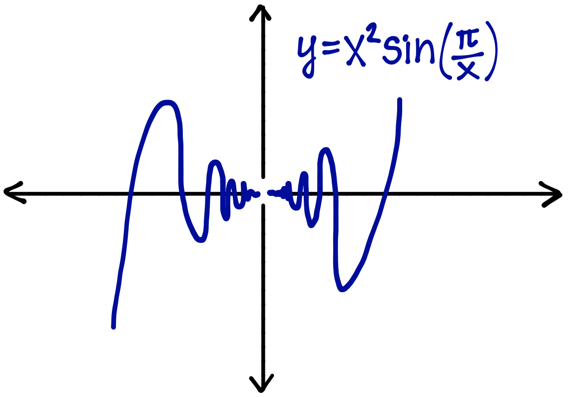
What about the limit $\lim_{x\to 0} x^2 \sin\left(\frac{\pi}{x}\right)$?

It would be wrong to use the Limit Law

$$\lim_{x\to 0} x^2 \sin\left(\frac{\pi}{x}\right) \neq \left[\lim_{x\to 0} x^2\right] \cdot \left[\underset{\text{DNE}}{\lim_{x\to 0} \sin\left(\frac{\pi}{x}\right)}\right]$$

because both limits must exist in order to apply that Law.

Graphically, it looks like limit is 0 as x→0
⋮
?



Like $\sin(\frac{\pi}{x})$, this function oscillates infinitely often as x→0, but, the amplitude of the sine waves keep shrinking because of x^2 factor

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The Squeeze Theorem.

Let f, g, and h be functions.

If $\boxed{f(x) \le g(x) \le h(x)}$ when x is *near* a, except possibly at a,

and $\boxed{\lim_{x\to a} f(x)} = L = \boxed{\lim_{x\to a} h(x)}$ for some unique real number L,

then

$$\lim_{x\to a} f(x) = L \le \lim_{x\to a} g(x) \le L = \lim_{x\to a} h(x)$$

$$\boxed{\therefore \lim_{x\to a} g(x) = L}$$

Going back to $\lim_{x\to 0} x^2 \sin\left(\frac{\pi}{x}\right)$

for all $x \in \mathbb{R}$, except $x=0$, we have

$$-1 \le \sin\left(\frac{\pi}{x}\right) \le 1$$

$$\therefore -x^2 \le x^2 \sin\left(\frac{\pi}{x}\right) \le x^2$$

Since $x^2 \ge 0$ for all $x \in \mathbb{R}$, these inequalities remain true when we multiply both sides by x^2

We also know $\lim_{x\to 0} -x^2 = 0 = \lim_{x\to 0} x^2$

By Squeeze Theorem,

$$\lim_{x\to 0} -x^2 = 0 \le \lim_{x\to 0} x^2 \sin\left(\frac{\pi}{x}\right) \le 0 = \lim_{x\to 0} x^2$$

$$\boxed{\therefore \lim_{x\to 0} x^2 \sin\left(\frac{\pi}{x}\right) = 0}$$

CONTINUITY

A function f is **continuous at a number** a if $\lim_{x\to a} f(x) = f(a)$

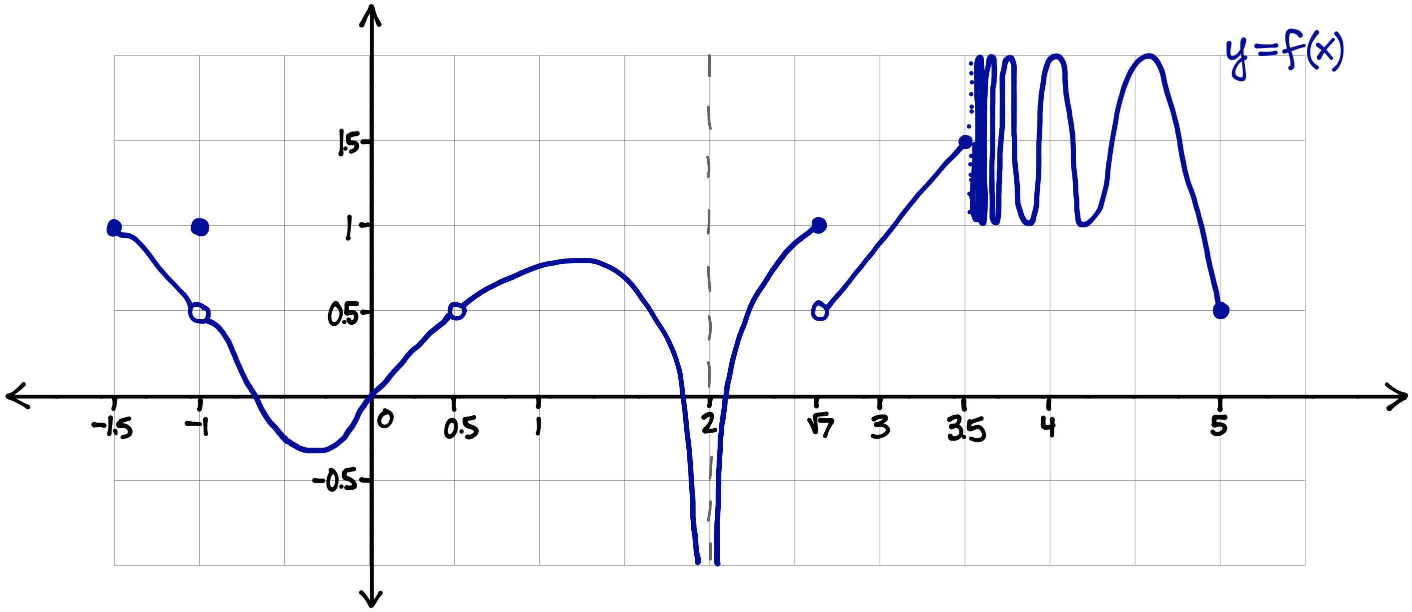
In order for $\lim_{x\to a} f(x) = f(a)$, three things must be true (by definition of this limit's existence):

1. ◇ $f(a)$ must be defined ($a \in$ Domain of f)
2. ◇ $\lim_{x\to a} f(x)$ must exist ($\Rightarrow \lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x)$ for some unique $L \in \mathbb{R}$)
3. ◇ $\lim_{x\to a} f(x) = f(a)$ (so the L from above must also equal the value of f at a)

If f is defined *near* a, but f fails to be continuous at a, then f is called **discontinuous at** a, or we say that f **has a discontinuity at** a.

REASONS WHY A FUNCTION COULD BE DISCONTINUOUS

Example 4.2. Consider the graph of f below.



a	Is f continuous at $x = a$? Explain why or why not.
$a = -1.5$	NO. $\lim_{x\to -1.5^-} f(x)$ DNE ∴ $\lim_{x\to 1.5} f(x)$ DNE (fails 2.)
$a = -1$	NO. $\lim_{x\to -1} f(x) = 0.5$ but $f(-1) = 1$. (fails 3.)
$a = 0$	YES. $\lim_{x\to 0} f(x) = 0 = f(0)$ ✓
$a = 0.5$	NO. $f(0.5)$ is undefined (fails 1.)
$a = 2$	NO. $\lim_{x\to 2} f(x) = -\infty$ (∴ DNE) (fails 2.)
$a = \sqrt{7}$	NO. $\lim_{x\to \sqrt{7}^-} f(x) \neq \lim_{x\to \sqrt{7}^+} f(x)$ ∴ $\lim_{x\to \sqrt{7}} f(x)$ DNE. (fails 2)
$a = 3.5$	NO. $\lim_{x\to 3.5^+} f(x)$ DNE ∴ $\lim_{x\to 3.5} f(x)$ DNE (fails 2)
$a = 4$	YES. $\lim_{x\to 4} f(x) = 2 = f(4)$ ✓
$a = 5$	NO. $\lim_{x\to 5^+} f(x)$ DNE ∴ $\lim_{x\to 5} f(x)$ DNE (fails 2)

Summary of possible reasons why f could be discontinuous at $x = a$.

- ○ a is not in the domain of f
  (*e.g.* hole, vertical asymptote)
- ○ limit of $f(x)$ as $x \to a$ DNE
  (*e.g.* infinite limit, no unique limit L, different one-sided limits, one-sided limit DNE)
- ○ limit of $f(x)$ as $x \to a$ exists, but isn't equal to $f(a)$
  (*e.g.* jump in the graph of f at $x = a$)

► For discontinuities, look for holes, jumps, and vertical asymptotes.

ONE-SIDED CONTINUITY

► A function $f(x)$ is **continuous...**

...from the left at a number a if

$$\lim_{x\to a^-} f(x) = f(a)$$

...from the right at a number a if

$$\lim_{x\to a^+} f(x) = f(a)$$

Example 4.3. Reconsider the function f given in Example 4.2.

a	Is f continuous at $x = a$ from the left, from the right, or neither? Explain.	
$a = -1.5$	NO. $\lim_{x\to -1.5^-} f(x)$ DNE.	YES. $\lim_{x\to -1.5^+} f(x) = 1 = f(-1.5)$ ✓
$a = -1$	NO. $\lim_{x\to -1^-} f(x) = 0.5$ but $f(-1) = 1$.	NO. $\lim_{x\to -1^+} f(x) = 0.5$ but $f(-1) = 1$.
$a = 0$	YES. $\lim_{x\to 0^-} f(x) = 0 = f(0)$ ✓	YES. $\lim_{x\to 0^+} f(x) = 0 = f(0)$ ✓
$a = 0.5$	NO. $f(0.5)$ is undefined	NO. $f(0.5)$ is undefined
$a = 2$	NO. $\lim_{x\to 2^-} f(x) = -\infty$ $\therefore$ DNE	NO. $\lim_{x\to 2^+} f(x) = -\infty$ $\therefore$ DNE
$a = \sqrt{7}$	YES. $\lim_{x\to \sqrt{7}^-} f(x) = 1 = f(\sqrt{7})$ ✓	NO. $\lim_{x\to \sqrt{7}^+} f(x) = 0.5$ but $f(\sqrt{7}) = 1$
$a = 3.5$	YES. $\lim_{x\to 3.5^-} f(x) = 1.5 = f(3.5)$ ✓	NO. $\lim_{x\to 3.5^+} f(x)$ DNE.
$a = 4$	YES. $\lim_{x\to 4^-} f(x) = 2 = f(4)$ ✓	YES. $\lim_{x\to 4^+} f(x) = 2 = f(4)$ ✓

Continuous on an Interval

- A function f is **continuous on an interval** if f is continuous at every number in the interval.
- If f is defined only on one side of an endpoint of the interval, we understand continuous at the endpoint to mean continuous from the right, or continuous from the left.
- Informally, f is **continuous on an interval** if we can trace the graph of f along the entire interval, without needing to lift our pencil off the paper.

Theorem 4.4. Let k be a constant.

If f and g are continuous at a number a, then the following functions are also continuous at a:

$f+g$ $\quad f-g$ $\quad kf$ $\quad fg$ $\quad \frac{f}{g} \leftarrow$ if $g(a) \neq 0$

Theorem 4.5. The following types of functions are continuous at every number in their domains:

- polynomials
- rational functions
- root functions
- trig functions
- inverse trig functions
- exponential and logarithmic functions

A Limit Law for Compositions of Continuous Functions

Theorem 4.6. Let f and g be functions.

If f is continuous at b and $\lim\limits_{x\to a} g(x) = b$, then $\lim\limits_{x\to a} f(g(x)) = f(b)$.

That is, $\boxed{\lim\limits_{x\to a} f(g(x)) = f\left(\lim\limits_{x\to a} g(x)\right)}$

Example 4.7. Evaluate $\lim_{x\to 1}\sin\left(\frac{\pi-\pi\sqrt{x}}{1-x}\right)$

Because sine is continuous, we can use the theorem!

$$\lim_{x\to1}\sin\left(\frac{\pi-\pi\sqrt{x}}{1-x}\right)$$

$$=\sin\left(\lim_{x\to1}\frac{\pi-\pi\sqrt{x}}{1-x}\right)$$

$$=\sin\left(\lim_{x\to1}\left(\frac{\pi(1-\sqrt{x})}{1-x}\right)\left(\frac{1+\sqrt{x}}{1+\sqrt{x}}\right)\right)$$

$$=\sin\left(\lim_{x\to1}\frac{\pi\cancel{(1-x)}}{\cancel{(1-x)}(1+\sqrt{x})}\right)$$

$$=\sin\left(\lim_{x\to1}\frac{\pi}{1+\sqrt{x}}\right)$$

$$=\sin\left(\frac{\pi}{2}\right)$$

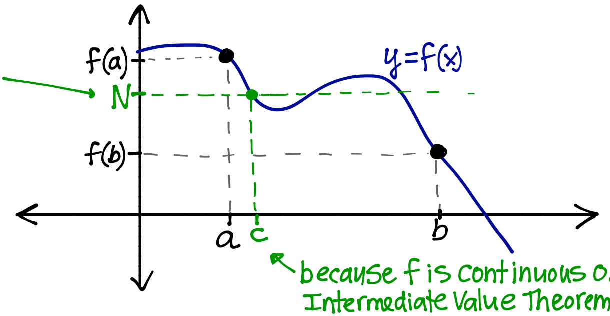
$$=1$$

INTERMEDIATE VALUE THEOREM

Suppose that f is continuous on the closed interval $[a,b]$ and $f(a)\neq f(b)$. If N is any number between $f(a)$ and $f(b)$, then

there exists a number $c\in(a,b)$ such that $f(c)=N$.

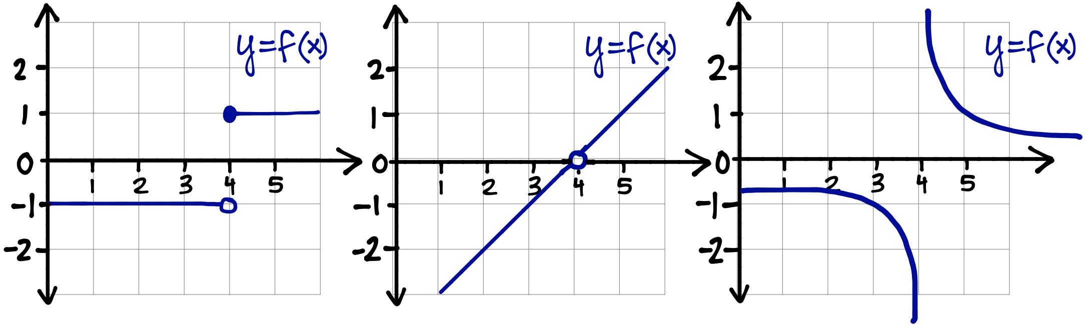
Pick some # N that is anywhere between f(a) and f(b)



because f is continuous on [a,b], the Intermediate Value Theorem guarantees that we can find a # $c\in(a,b)$ such that $f(c)=L$ (in the picture there were 3 different #s that could be the c for this function. Regardless, I.V.T. guarantees at least one such # c)

The Intermediate Value Theorem may seem obvious, but don't forget that it relies on the fact that f is **continuous** on the interval $[a, b]$.

Exercise 4.8. Think about the ways in which a function can have a discontinuity. Then draw several possibilities in which a function f has the property that $f(3) = -1$, $f(5) = 1$, but there is no point $c \in [3, 5]$ such that $f(c) = 0$.



Example 4.9. Use the **Intermediate Value Theorem** to prove that the equation

$$x^5 - x^4 + x^3 - x - 1 = 0$$

has a root in the interval $[1, 2]$.

- $f(x) = x^5 - x^4 + x^3 - x - 1$ is a polynomial $\therefore$ $f(x)$ is continuous at all real #s
- In particular, f is continuous on $[1, 2]$
- $f(1) = 1^5 - 1^4 + 1^3 - 1 - 1 = -1$
- $f(2) = 2^5 - 2^4 + 2^3 - 2 - 1 = 21$

$\Big\}$ Choose $N = 0$, and note that $f(1) = -1 < 0 < 21 = f(2)$

$\therefore$ By I.V.T., there exists some # $c \in (1, 2)$ such that $f(c) = N = 0$.

$\Rightarrow$ f has a root in $(1, 2)$

Limits at Infinity & Horizontal Asymptotes

- Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = L$$

  means that the values of $f(x)$ can be made arbitrarily close to a unique real number L so long as x is sufficiently large.

- Let f be a function defined on some interval $(-\infty, a)$. Then

$$\lim_{x \to -\infty} f(x) = L$$

  means that the values of $f(x)$ can be made arbitrarily close to a unique real number L so long as x is a sufficiently large negative number.

- The line $y = L$ is called a **horizontal asymptote** if $\lim\limits_{x \to -\infty} f(x) = L$ or $\lim\limits_{x \to \infty} f(x) = L$.

Useful Fact (Theorem)

- If $r > 0$ is a rational number, then $\displaystyle\lim_{x\to\infty} \frac{1}{x^r} = 0$

  Ex $\displaystyle\lim_{x\to\infty} \frac{1}{\sqrt{x}} = 0$

- If $r > 0$ is a rational number such that x^r is defined for all $x \in \mathbb{R}$, then

$$\lim_{x\to-\infty} \frac{1}{x^r} = 0$$

Ex $\displaystyle\lim_{x\to-\infty} \frac{1}{\sqrt{x}}$ DNE

↖ $\sqrt{x}$ only defined for $x \geq 0$

Example 4.10. $\displaystyle\lim_{x\to\infty} \frac{8x^3 - x^2}{1 + x - x^3}$

$$= \lim_{x\to\infty} \left(\frac{8x^3 - x^2}{1 + x - x^3}\right)\left(\frac{\frac{1}{x^3}}{\frac{1}{x^3}}\right)$$

$$= \lim_{x\to\infty} \frac{8 - \frac{1}{x}}{\frac{1}{x^3} + \frac{1}{x^2} - 1}$$

$$= \frac{8 - 0}{0 + 0 - 1} = -8$$

Example 4.11. $\displaystyle\lim_{x\to-\infty} \frac{x}{|x|}$

Since $x \to -\infty$, we may assume $x < 0$ $\therefore$ $|x| = -x$

$$= \lim_{x\to-\infty} \frac{x}{-x}$$

$$= \lim_{x\to-\infty} -1$$

$$= -1$$

Example 4.12. $\displaystyle\lim_{x\to\infty} \frac{\sin(x)}{x}$

for all x, $-1 \leq \sin(x) \leq 1$

for all $x > 0$, $\frac{1}{x} > 0$ $\therefore$ $-1\left(\frac{1}{x}\right) \leq (\sin x)\left(\frac{1}{x}\right) \leq 1\left(\frac{1}{x}\right)$

$$\Rightarrow \quad -\frac{1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x}$$

It's also true that $\displaystyle\lim_{x\to\infty} -\frac{1}{x} = 0 = \lim_{x\to\infty} \frac{1}{x}$

By the Squeeze Theorem, $\displaystyle\lim_{x\to\infty} \frac{\sin x}{x} = 0$

Example 4.13. $\lim\limits_{x\to\infty} \sqrt{9x^2+x}-3x$ indeterminate of the form $\infty-\infty$

$$= \lim_{x\to\infty}\left(\sqrt{9x^2+x}-3x\right)\left(\frac{\sqrt{9x^2+x}+3x}{\sqrt{9x^2+x}+3x}\right)$$

$$= \lim_{x\to\infty} \frac{x}{\sqrt{9x^2+x}+3x}$$

$$= \lim_{x\to\infty}\left(\frac{x}{\sqrt{9x^2+x}+3x}\right)\left(\frac{\frac{1}{x}}{\frac{1}{x}}\right)$$

$$= \lim_{x\to\infty}\left(\frac{1}{\sqrt{\frac{1}{x^2}}\left(\sqrt{9x^2+x}\right)+3}\right)$$

as $x\to\infty$, $x>0$. $\therefore \sqrt{\frac{1}{x^2}} = \frac{1}{x}$. Not true if $x<0$!

$$= \lim_{x\to\infty}\frac{1}{\sqrt{9+\frac{x}{x^2}}+3}$$

$$= \lim_{x\to\infty}\frac{1}{\sqrt{9+\frac{1}{x}}+3}$$

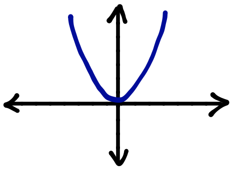
$$= \frac{1}{\sqrt{9+0}+3}$$

$$= \frac{1}{6}$$

Exercise

$$\lim_{x\to-\infty}\sqrt{9x^2+x}-3x$$

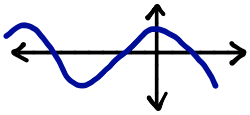
Example 4.14. $\lim\limits_{x\to\infty} x^2 = \infty$ (DNE)



an infinite limit at infinity

$\therefore$ x^2 gets arbitrarily large as $x\to\infty$

Example 4.15. $\lim\limits_{x\to-\infty} \cos(x)$ DNE.



cos(x) <u>does not</u> become arbitrarily close to a unique real # L, as $x\to-\infty$

STUDY GUIDE

Important terms and concepts:

- ◇ **The Squeeze Theorem**
- ◇ **Continuity** continuous at $x=a$ continuous on an interval
- ◇ **Discontinuity** hole jump vertical asymptote
- ◇ **Intermediate Value Theorem**
- ◇ **Limits At Infinity & Horizontal Asymptotes**

Exercises: Stewart, 8th ed. §2.5, pg. 124 # 17, 19, 21, 35, 37, 39, 45
§2.6 pg. 137 # 3, 7, 9, 15, 17, 19, 23, 27, 29, 31, 33, 35, 47, 49, 57