# Practice Problems on Fourier Series

It may be useful for your work to recall the following integrals :

$$\int u\cos u\,du = \cos u + u\sin u + C;$$

$$\int u\sin u\,du = \sin u - u\cos u + C;$$

$$\int_{-\pi}^{\pi} \cos mx\cos nx\,dx = \begin{cases} 0, & when\ m \neq n, \\ \pi, & when\ m = n. \end{cases}$$

$$\int_{-\pi}^{\pi} \sin mx\sin nx\,dx = \begin{cases} 0, & when\ m \neq n, \\ \pi, & when\ m = n. \end{cases}$$

$$\int_{-\pi}^{\pi} \cos mx\sin nx\,dx = 0 \text{ for all } m \text{ and } n.$$

**Problem 1.** Find the period of the given periodic function:

| | | |
|---|---|---|
| $(a)$ $\cos 2x$ | $(b)$ $\sin 3\pi x$ | $(c)$ $\sin \frac{\pi x}{3}$ |
| $(d)$ $\cot 3x$ | $(e)$ $3\sin 5x$ | $(f)$ $3\sin x + \cos 2x$ |
| $(g)$ $5\cos 3x + 2\cos 2x$ | $(h)$ $5\cos\frac{x}{3} + 2\cos 2x$ | $(i)$ $\lvert cosx \rvert$ |
| $(j)$ $\cos^2 x$ | $(k)$ $\cos^3 x$ | $(l)$ $\cos^4 x$ |

**Problem 2.** For a given $2\pi$-periodic function,
(i) sketch several periods of its graph;
(ii) find its Fourier series.

(a) $f(x) = 3, \quad -\pi < x \le \pi.$

(b) $f(x) = \begin{cases} 2, & -\pi < x < 0, \\ -1, & 0 < x < \pi, \\ \frac{1}{2}, & x = -\pi,\ 0,\ \pi. \end{cases}$

(c) $f(x) = x, \quad -\pi \le x < \pi;$
Redefine $f(x)$ to the average value at the points of discontinuity.

(d) $f(x) = \cos x, \ -\pi < x \le \pi.$

(e) $f(x) = \begin{cases} 2\pi + x, & -\pi \le x < 0, \\ 0, & 0 \le x < \pi. \end{cases}$

Redefine $f(x)$ to the average value at the points of discontinuity.

**Problem 3.** For a $2L$-periodic function given on one full period,
(i) define $f(x)$ at each point of discontinuity by the average value;
(ii) find the Fourier series of $f$.

(a) $f(x) = \begin{cases} 3, & -2 < x < 0, \\ -1, & 0 < x < 2. \end{cases}$

(b) $f(x) = \begin{cases} 0, & 0 < x < 1, \\ 1, & 1 < x < 2. \end{cases}$

(c) $f(x) = \begin{cases} x, & -2\pi < x < 0, \\ -1, & 0 < x < 2\pi. \end{cases}$

**Problem 4.** For a $2L$-periodic function given on one full period,
(i) define $f(x)$ at each point of discontinuity by the average value;
(ii) find the Fourier coefficient $a_0$.

(a) $f(x) = \begin{cases} 5, & -1 < x < 0, \\ -1, & 0 < x < 2. \end{cases}$

**Problem 5.** For the function $f(x)$ defined on the half-period $0 < x < L$,
(i) give the **even** extension to the full period $-L < x < L$;
(ii) for the functions in (a) and (b), find the Fourier **cosine** series.

(a) $f(x) = -1, \quad 0 < x < 1.$

(b) $f(x) = 1 - x, \quad 0 < x < 2.$

(c) $f(x) = \sin x, \quad 0 < x < \pi.$

(d) $f(x) = \begin{cases} x - 5, & 0 < x < 1, \\ 0, & 1 < x < 2. \end{cases}$

(e) $f(x) = \begin{cases} 0, & 0 < x < 1, \\ 1 - x, & 1 < x < 2. \end{cases}$

**Problem 6.** For the functions in **Problem 5**,
(i) give the odd extension to the full period $-L < x < L$;
(ii) for the functions in (a) and (b), find the Fourier **sine** series.

## Answers

**Problem 1.**
(a) $\pi$. (b) $\frac{2}{3}$. (c) 6. (d) $\frac{\pi}{3}$. (e) $\frac{2\pi}{5}$. (f) $2\pi$. (g) $2\pi$. (h) $6\pi$. (i) $\pi$. (j) $\pi$.
(k) $2\pi$. (l) $\pi$.

**Problem 2.**
**(a)** $a_0 = 6,\ a_n = b_n = 0$ for $n \geq 1$.
**(b)** $a_0 = 1,\ a_n = 0,\ b_n = -\dfrac{6}{n\pi}$ for $n$ odd, 0 for $n$ even, and
$$f(x) = \frac{1}{2} - \frac{6}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \dots\right).$$
**(c)** $f_{av}(\pi) = 0,\ f(x) = 2\left(\sin x - \dfrac{1}{2}\sin 2x + \dfrac{1}{3}\sin 3x - \dots\right)$.
**(d)** $a_0 = 0,\ a_1 = 1,\ a_n = 0$ for $n \geq 2$; $b_n = 0$.
**(e)** For $x = \pm 2n\pi,\ (n = 0, 1, \dots),\ f_{av} = \pi$;
for $x = \pm(2n+1)\pi,\ (n = 0, 1, \dots),\ f_{av} = \dfrac{\pi}{2}$; $a_0 = \dfrac{3\pi}{2},\ a_n = \dfrac{2}{n^2\pi}$ for $n$ odd, 0 for $n$ even;
$b_n = -\dfrac{3}{n}$ for $n$ odd, $-\dfrac{1}{n}$ for $n$ even; and
$$f(x) = \frac{3\pi}{4} + \frac{2}{\pi}\cos x - 3\sin x - \frac{1}{2}\sin 2x + \frac{2}{9\pi}\cos 3x - \sin 3x - \frac{1}{4}\sin 4x + \dots.$$

**Problem 3.**
**(a)** $f_{av}(0) = f_{av}(2) = 1,\ a_0 = 2,\ a_n = 0$ for $n \geq 1$;
$b_n = -\dfrac{8}{n\pi}$ for $n$ odd, 0 for $n$ even; and
$$f(x) = 1 - \frac{8}{\pi}\left(\sin\frac{\pi x}{2} + \frac{1}{3}\sin\frac{3\pi x}{2} + \frac{1}{5}\sin\frac{5\pi x}{2} + \dots\right).$$
**(b)** $f_{av}(0) = f_{av}(1) = 0.5,\ a_0 = 1,\ a_n = 0$ for $n \geq 1$;
$b_n = -\dfrac{2}{n\pi}$ for $n$ odd, 0 for $n$ even; and
$$f(x) = \frac{1}{2} - \frac{2}{\pi}\left(\sin \pi x + \frac{1}{3}\sin 3\pi x + \frac{1}{5}\sin 5\pi x + \dots\right).$$
**(c)** $f_{av}(0) = -0.5, f_{av}(2\pi) = \frac{-2\pi-1}{2},\ a_0 = -\pi - 1,\ a_n = \dfrac{4}{n^2\pi}$ for $n$ odd, 0 for $n$ even;

$b_n = \frac{2}{n} - \frac{2}{n\pi}$ for $n$ odd, $-\frac{2}{n}$ for $n$ even; and

$f(x) = \frac{-\pi - 1}{2} + \frac{4}{\pi}\cos\frac{x}{2} + (2 - \frac{2}{\pi})\sin\frac{x}{2} - \sin x + \frac{4}{9\pi}\cos\frac{3x}{2} + (\frac{2}{3} - \frac{2}{3\pi})\sin\frac{3x}{2} -$
$- \frac{1}{2}\sin 2x + \ldots.$

**Problem 4.**

**(a)** $f_{av}(3n) = f_{av}(3n+2) = 2,$ for integer $n$; $a_0 = 2.$

**Problem 5.**

**(a)** $f(x) = -1$ for $-1 < x < 0$; $a_0 = -2,\ a_n = 0$ for $n \geq 1.$

**(b)** $f(x) = 1 + x$ for $-2 < x < 0$; $a_0 = 0,\ a_n = \frac{8}{n^2\pi^2}$ for $n$ odd, $0$ for $n$ even.

**(c)** $f(x) = -\sin x$ for $-\pi < x < 0.$

**(d)** $f(x) = -x - 5$ for $-1 < x < 0$; $0$ for $-2 < x < -1.$

**(e)** $f(x) = 0$ for $-1 < x < 0$; $1 + x$ for $-2 < x < -1.$

**Problem 6.**

**(a)** $f(x) = 1$ for $-1 < x < 0$; $b_n = -\frac{4}{n\pi}$ for $n$ odd, $0$ for $n$ even.

**(b)** $f(x) = -1 - x$ for $-2 < x < 0$; $b_n = 0$ for $n$ odd, $\frac{4}{n\pi}$ for $n$ even.

**(c)** $f(x) = \sin x$ for $-\pi < x < 0.$

**(d)** $f(x) = x + 5$ for $-1 < x < 0$; $0$ for $-2 < x < -1.$

**(e)** $f(x) = 0$ for $-1 < x < 0$; $-1 - x$ for $-2 < x < -1.$

**Problem 7.** Consider the $2\pi$ periodic function given on interval $(-\pi, \pi)$ by the formulas:

$$f(t) = \begin{cases} 1, -\pi < t \le -\frac{\pi}{2} \\ -1, -\frac{\pi}{2} < t \le \frac{\pi}{2} \\ 1, \frac{\pi}{2} < t \le \pi \end{cases}$$

The Fourier Series of this function is given to be

$$f(t) = \sum_{k=0}^{\infty} \frac{4(-1)^{k+1}}{\pi(2k+1)} \cos((2k+1)t).$$

(a) Determine the antiderivative $F(t)$ for $f(t)$ with $F(0) = 0$. Sketch the graphs of $f(t)$ and $F(t)$.

(b) Use the integration theorem to find the Fourier series for $F(t)$.

(c) Use the integration theorem again to find the Fourier series for a second antiderivative of $f(t)$.

(d) What is the condition(s) when the integration theorem is applicable?

**Problem 8.** Consider the $2\pi$ periodic function given on interval $(-\pi, \pi)$ by the formula $f(t) = e^t + e^{-t}$.

The Fourier series of $f(t)$ is given to be:

$$f(t) = \frac{e^{\pi} - e^{-\pi}}{\pi} \left( 1 + \sum_{k=1}^{\infty} \frac{2(-1)^k}{1+k^2} \cos(kt) \right).$$

(a) Use the differentiation theorem to find the Fourier series for $f'(t) = e^t - e^{-t}$.

(b) Sketch both $f(t)$ and $f'(t)$.

(c) Can one use the differentiation theorem to find the Fourier series of $f''(t)$? Why?

**Answers.**

**Problem 7.** (a)

$$F(t) = \begin{cases} \pi + t, -\pi < t \le -\frac{\pi}{2} \\ -t, -\frac{\pi}{2} < t \le \frac{\pi}{2} \\ t - \pi, \frac{\pi}{2} < t \le \pi \end{cases}$$

(b)

$$F(t) = \sum_{k=0}^{\infty} \frac{4(-1)^{k+1}}{\pi(2k+1)^2} \sin((2k+1)t).$$

(c)

$$\text{second antiderivative of } f(t) = C + \sum_{k=0}^{\infty} \frac{4(-1)^k}{\pi(2k+1)^3} \cos((2k+1)t).$$

**Problem 8.** (a)

$$f'(t) = \frac{e^{\pi} - e^{-\pi}}{\pi} \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}k}{1+k^2} \sin(kt).$$

(c) No, since $f'(t)$ is discontinuous at points $\pi + 2\pi n,\ n = .., -1, 0, 1, 2, \ldots$.