MATH2004 C — Test 3: 11:35 am - 12:25 pm, Nov 16

Name and Student Number:

Total points: 20.

Closed book! Non-programmer calculators are allowed!

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1. (3 points) Find the critical point(s) of $f(x, y) = x^3 - 3xy + y^3$.

   Solution: The first partial derivatives are $f_x(x, y) = 3x^2 - 3y$, $f_y(x, y) = -3x + 3y^2$.

   Setting $f_x = 0$ and $f_y = 0$: $3x^2 - 3y = 0$, $-3x + 3y^2 = 0$.

   We imply that $(x, y) = (0, 0), (1, 1)$.

2. (9 points=3+2+2+2) Let $f(x, y) = x^3 + 2y^3 - 3x^2 - 3y^2 + 1$.

   (i) Calculate $f_x(x, y)$, $f_y(x, y)$, $f_{xx}(x, y)$, $f_{xy}(x, y)$, $f_{yy}(x, y)$.

   (ii) Find the directional derivative of $f(x, y)$ at the point $(1, -1)$ in the direction of $\mathbf{v} =< -6, -8 >$.

   (iii) Given that $(2, 1)$ is a critical point of $f(x, y)$, determine if it is a local maximum, a local minimum or a saddle-point.

   (iv) Let $x = r \cos t$, $y = r \sin t$. Use chain rule to find $f_t(x, y)$ or $\frac{\partial f}{\partial t}$ (you don't need to simplify the result).

   Solution: (i)

   $$f_x(x, y) = 3x^2 - 6x, f_y(x, y) = 6y^2 - 6y.$$

   $$f_{xx}(x, y) = 6x - 6, \qquad f_{xy}(x, y) = 0, \qquad f_{yy}(x, y) = 12y - 6.$$

   (ii) $|\mathbf{v}| = | < -6, -8 > | = 10$, $\mathbf{u} =< -3/5, -4/5 >$. $f_x(1, -1) = -3, f_y(1, -1) = 12$.

   $$D_{\mathbf{v}} f(1, -1) = -3(-3/5) + 12(-4/5) = -39/5.$$

   (iii) Note that $D(x, y) = (6x - 6)(12y - 6)$.

   $D(2, 1) = 36 > 0$,, $f_{xx}(2, 1) = 6 > 0$, so (2,1) is a local minimum point.

   (iv) $\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} = (3x^2 - 6x)(-r \sin t) + (6y^2 - 6y) r \cos t$.

3. (8 points=2+2+4) Let $f(x,y,z) = 4x + 2z$, $g(x,y,z) = x^2 + 2y^2 + z^2$.

(i) Calculate $\bigtriangledown f(x,y,z)$ and $\bigtriangledown g(x,y,z)$.

(ii) Find the tangent plane at the point $(1,1,2)$ on the surface $x^2 + 2y^2 + z^2 = 7$.

(iii) Use the method of Lagrange multipliers to find the maximum and minimum values of $f(x,y,z)$ subject to the constraint $g(x,y,z) = x^2 + 2y^2 + z^2 = 5$.

Solution: (i)

$$\bigtriangledown f(x,y,z) =< 4,0,2 >, \quad \bigtriangledown g(x,y,z) =< 2x, 4y, 2z > .$$

(ii) At $(1,1,2)$, $\bigtriangledown g(1,1,2) =< 2,4,4 >$. The equation of the tangent plane is

$$2(x-1) + 4(y-1) + 4(z-2) = 0. \quad 2x + 4y + 4z - 14 = 0.$$

(iii) We solve

$$\bigtriangledown f - \lambda \bigtriangledown g = 0 \quad (\textbf{1 point for writing this equation.}) \tag{1}$$
$$g(x,y) = 5. \tag{2}$$

The equations (1)-(2) are equivalent to

$$4 - 2\lambda x = 0 \tag{3}$$
$$0 - 4\lambda y = 0 \tag{4}$$
$$2 - 2\lambda z = 0 \tag{5}$$
$$x^2 + 2y^2 + z^2 = 5. \tag{6}$$

So we have

$$x = 2/\lambda, \qquad y = 0, \qquad z = 1/\lambda.$$

Hence

$$\frac{4}{\lambda^2} + 0 + \frac{1}{\lambda^2} = 5,$$

which gives $\lambda = \pm 1$.

When $\lambda = 1$, we have $x = 2$, $y = 0$, and $z = 1$.

When $\lambda = -1$, we have $x = -2$, $y = 0$, and $z = -1$.

The maximum is $f(2,0,1) = 10$.

The minimum is $f(-2,0,-1) = -10$.