MATH2004 C — Test 2: 11:35 am - 12:25 pm, Oct. 26

Name and Student Number:

Total points: 20. No partial marks for Questions 1-4.

Closed book! Non-programmer calculators are allowed!

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1. (1.5 marks) Find the slope of the equation of the tangent line to $r = 4\cos\theta$ at $\theta = \pi/3$.

   (a) 1 (b) 2 (c) 3 (d) $\sqrt{3}$ (e) $\frac{1}{\sqrt{3}}$

   Solution: (e)

$$\frac{dy}{dx} = \frac{r'_\theta \sin\theta + r\cos\theta}{r'_\theta \cos\theta - r\sin\theta} = \frac{-\sin^2\theta + \cos^2\theta}{-2\sin\theta\cos\theta} = \frac{1}{\sqrt{3}}.$$

2. (1.5 marks) Given the linear equation of the plane Π: $2x + 3y + 4z - 4 = 0$ and the line $L : \mathbf{r} = (1, 2, 3) + t(3, 2, -1)$, find the intersection between the plane Π and the line L.

   (a) $(5, -2, 5)$ (b) $(-5, 2, 5)$ (c) $(-5, -2, -5)$ (d) $(5, 2, 5)$ (e) $(-5, -2, 5)$

   Solution: (e)

   The parametric equation of the line is:

$$x = 1 + 3t, y = 2 + 2t, z = 3 - t.$$

   Substitute this into the plane we get

$$2(1 + 3t) + 3(2 + 2t) + 4(3 - t) - 4 = 0, \Rightarrow t = -2.$$

   Thus the point is (-5,-2,5).

3. (1.5 marks) Let $\mathbf{r}(t) = (6t, 8t, te^t) = 6t\mathbf{i} + 8t\mathbf{j} + te^t\mathbf{k}$. Calculate $\int_0^1 \mathbf{r}(t)dt$.

   (a) $(3, 4, e)$ (b) $(3, 4, 1)$ (c) $(6, 8, e)$ (d) $(6, 8, 1)$ (e) $(3, 4, -e)$

   Solution: (b).

$$\int_0^1 \mathbf{r}(t)dt = (\int_0^1 6tdt, \int_0^1 8tdt, \int_0^1 te^t dt) = (3, 4, 1).$$

4. (2.5 marks) Let $\mathbf{r}(t) = (4t-1, t^2, t^3-3t) = (4t-1)\mathbf{i}+t^2\mathbf{j}+(t^3-3t)\mathbf{k}$. Find the curvature $\kappa(0)$ of $\mathbf{r}(t)$ at $t=0$.

Solution:

$\mathbf{r}'(t) = (4, 2t, 3t^2-3)$, $\mathbf{r}''(t) = (0, 2, 6t)$. At $t=0$, $\mathbf{r}'(0) = (4, 0, -3)$, $\mathbf{r}''(0) = (0, 2, 0)$, $\mathbf{r}'(0) \times \mathbf{r}''(0) = (6, 0, 8)$. Hence

$|\mathbf{r}'(0)| = 5$, $|\mathbf{r}'(0) \times \mathbf{r}''(0)| = 10$.

The curvature at $t = 0$ is:

$$\kappa(0) = \frac{|\mathbf{r}'(0) \times \mathbf{r}''(0)|}{|\mathbf{r}'(0)|^3} = \frac{10}{5^3} = \frac{2}{25}.$$

5. (6 points) Let Π be the plane passing through the three points $P(1, 2, -1)$, $Q(3, 1, -1)$ and $R(1, 1, 0)$.

   (i) (4 points) Find the linear equation of the plane Π.

   (ii) (2 points) Find the distance from the point $S(8, 1, 1)$ to the plane Π.

Solution: (i) We have

$$\vec{PQ} = < 2, -1, 0 >, \quad \vec{PR} = < 0, -1, 1 > .$$

The normal vector of the plane is

$$\mathbf{n} = \vec{PQ} \times \vec{PR} = < -1, -2, -2 > .$$

The equation of the plane is

$$< -1, -2, -2 > \cdot (\mathbf{r} - < 1, 2, -1 >) = 0, \Rightarrow -(x-1) - 2(y-2) - 2(z+1) = 0,$$

i.e.,

$$x + 2y + 2z - 3 = 0.$$

(ii). The distance is:

$$d = \frac{|8 + 2(1) + 2(1) - 3|}{\sqrt{1^2 + 2^2 + 2^2}} = \frac{9}{3} = 3.$$

6. (7 points) Let C be $\mathbf{r}(t) = (3t, \cos(4t), \sin(4t)) = 3t\mathbf{i} + \cos(4t)\mathbf{j} + \sin(4t)\mathbf{k}$.

(i)(2 points) Find the length of the curve from $t = 0$ to $t = 2$.

(ii) (5 points = 2+3) Find the unit tangent vector $\mathbf{T}(t)$ and the unit normal vector $\mathbf{N}(t)$.

Solution: (i) $\mathbf{r}'(t) =< 3, -4\sin(4t), 4\cos(4t) >$, which gives $|\mathbf{r}'(t)| = 5$.
So the length is

$$L = \int_0^2 |\mathbf{r}'(t)|dt = \int_0^2 5dt = 10.$$

(ii) From (i) we have

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} =< 3/5, -4/5\sin(4t), 4/5\cos(4t) > .$$

$$\mathbf{T}'(t) =< 0, -16/5\cos(4t), -16/5\sin(4t) >, \quad |\mathbf{T}'(t)| = 16/5.$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} =< 0, -\cos(4t), -\sin(4t) > .$$