EXAMPLES OF VARIABLES SEPARABLE EQUATIONS USING HOMOGENEOUS FUNCTIONS

Example 1. Solve

$$\frac{dy}{dx} = \frac{x\cos y}{e^x \sin y} \qquad y \neq n\pi \quad n = 0, \pm 1, \pm 2, \ldots.$$

Solution:

$$\frac{\sin y}{\cos y} dy = x\, e^{-x}\, dx$$

$$\int \frac{\sin y}{\cos y} dy = \int x\, e^{-x}\, dx + C$$

$$\int \frac{\sin y}{\cos y} dy \qquad \text{let} \qquad u = \cos y \quad \rightarrow \quad du = -\sin y\, dy \quad \rightarrow \quad dy = -\frac{du}{\sin y}$$

$$\int \frac{\sin y}{\cos y} dy = -\int \frac{\sin y}{u} \frac{du}{\sin y} = -\int \frac{du}{u} = -\ln(\cos y)$$

$$\int x\, e^{-x}\, dx$$

Let

$$u = x \qquad dv = e^{-x} dx$$
$$du = dx \qquad v = -e^{-x}$$

$$\int x\, e^{-x}\, dx = -x\, e^{-x} - \int \left(-e^{-x}\, dx\right) = -x\, e^{-x} - e^{-x} = -(x+1)e^{-x}$$

$$-\ln(\cos y) = -(x+1)e^{-x} + C$$

Example 2. Provide the explicit solution of the following differential equation

$$\left(1+x^2\right)\left(1+y^2\right)dx - xy\,dy = 0$$

Solution:

$$\frac{\left(1+x^2\right)dx}{x} = \frac{y\,dy}{1+y^2}$$

$$\int \frac{\left(1+x^2\right)dx}{x} = \int\left(\frac{1}{x}+x\right)dx = \ln x + \frac{x^2}{2}$$

$$\int \frac{y\,dy}{1+y^2} \quad \text{let } u = 1+y^2 \quad \rightarrow \quad du = 2\,y\,dy \quad \rightarrow \quad dy = \frac{1}{2\,y}\,du$$

$$\int \frac{y\,dy}{1+y^2} = \int \frac{y}{u}\,\frac{1}{2\,y}\,du = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln(u) = \frac{1}{2}\ln\left(1+y^2\right)$$

$$\ln x + \frac{x^2}{2} = \frac{1}{2}\ln\left(1+y^2\right) + C \rightarrow 2\ln x + x^2 - 2\,C = \ln\left(1+y^2\right)$$

$$2\ln x + x^2 - 2\,C = \ln\left(1+y^2\right) \rightarrow \ln x^2 + x^2 - 2\,C = \ln\left(1+y^2\right)$$

$$e^{\ln x^2 + x^2 - 2C} = e^{\ln\left(1+y^2\right)} \quad \rightarrow \quad x^2 e^{x^2} e^{-2C} = 1 + y^2$$

$$y = \pm\sqrt{C'\,x^2 e^{x^2} - 1}$$

Example 3. Find the explicit solution of the differential equation

$$\left(1+x^2\right)\frac{dy}{dx} + 2xy = x$$

Solution:

$$\left(1+x^2\right) dy + 2xy\, dx = x\, dx \rightarrow \left(1+x^2\right) dy = x\, dx - 2xy\, dx$$

$$\rightarrow \left(1+x^2\right) dy = x\left(1-2y\right) dx$$

$$\frac{dy}{1-2y} = \frac{x\, dx}{\left(1+x^2\right)}$$

$$\int \frac{dy}{1-2y} = \int \frac{x\, dx}{\left(1+x^2\right)}$$

$$\int \frac{dy}{1-2y} \quad \text{let} \quad u = 1-2y \quad \rightarrow du = -2\, dy \quad \rightarrow \quad dy = -\frac{1}{2}\, du$$

$$\int \frac{dy}{1-2y} = -\frac{1}{2}\int \frac{du}{u} = -\frac{1}{2}\ln\left(1-2y\right)$$

$$\int \frac{x\, dx}{\left(1+x^2\right)} \quad \text{let} \quad u = 1+x^2 \quad \rightarrow du = 2\, x\, dx \quad \rightarrow \quad dx = \frac{1}{2\, x}\, du$$

$$\int \frac{x\, dx}{\left(1+x^2\right)} = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln\left(1+x^2\right)$$

$$-\frac{1}{2}\ln\left(1-2y\right) = \frac{1}{2}\ln\left(1+x^2\right) + C \quad \rightarrow \quad -\ln\left(1-2y\right) = \ln\left(1+x^2\right) + 2C$$

$$\ln\left(1-2y\right)^{-1} = \ln\left(1+x^2\right) + 2C \rightarrow \frac{1}{1-2y} = C'\left(1+x^2\right) \rightarrow \frac{C''}{1+x^2} = 1-2y$$

$$-\frac{C''}{1+x^2} = -1+2y \quad \rightarrow \quad y = \frac{1}{2}\left(1-\frac{C''}{1+x^2}\right)$$

EXAMPLES OF VARIABLES SEPARABLE EQUATIONS USING HOMOGENEOUS FUNCTIONS

For example the function

$$f(x, y) = x^4 + 8x^3y - 2x^2y^2 + y^4$$

is homogeneous of degree 4 because

$$f(sx, sy) = (sx)^4 + 8(sx)^3(sy) - 2(sx)^2(sy)^2 + (sy)^4 =$$

$$\text{degree} \downarrow$$

$$s^4\left(x^4 + 8x^3y - 2x^2y^2 + y^4\right) = s^4 f(x, y)$$

Example 4. Solve the differential equation

$$\overset{P}{\left(y - \sqrt{x^2 + y^2}\right)}dx \overset{Q}{- x}\, dy = 0$$

Since P and Q are homogeneous functions of the same degree (1), let

$$y(x) = x\,v(x) \qquad \rightarrow \qquad dy = vdx + x\,dv$$

$$\left(xv - \sqrt{x^2 + (xv)^2}\right)dx - x\left(vdx + x\,dv\right) = 0$$

$$x\left(v - \sqrt{1 + v^2}\right)dx - x\left(vdx + x\,dv\right) = 0$$

$$\left(v - \sqrt{1 + v^2}\right)dx - \left(vdx + x\,dv\right) = 0$$

$$\left(v - \sqrt{1 + v^2} - v\right)dx - x\,dv = 0$$

$$\frac{dx}{x} = -\frac{dv}{\sqrt{1 + v^2}}$$

$$\int \frac{dx}{x} = -\int \frac{dv}{\sqrt{1+v^2}} + C$$

$$\ln x = -\int \frac{dv}{\sqrt{1+v^2}} + C \quad \rightarrow \quad \ln x = -\ln\left(v+\sqrt{1+v^2}\right) + C$$

$$\ln x = -\ln\left(\frac{y}{x}+\sqrt{1+\left(\frac{y}{x}\right)^2}\right) + C$$

$$x = \frac{C'}{\frac{y}{x}+\sqrt{1+\left(\frac{y}{x}\right)^2}}$$

Example 5. Solve the differential equation

$$\overbrace{\left(\frac{1}{x}-\frac{y}{x^2}e^{y/x}\right)}^{P}dx+\overbrace{\left(\frac{1}{x}e^{y/x}-\frac{1}{y}\right)}^{Q}dy=0$$

Since P and Q are homogeneous functions of the same degree (-1), let

$$y(x) = x\,v(x) \quad \rightarrow \quad dy = vdx + x\,dv$$

$$\left(\frac{1}{x}-\frac{x\,v}{x^2}e^{x\,v/x}\right)dx+\left(\frac{1}{x}e^{x\,v/x}-\frac{1}{x\,v}\right)dy=0$$

$$\frac{1}{x}\left(1-v\,e^v\right)dx+\frac{1}{x}\left(e^v-\frac{1}{v}\right)dy=0$$

$$\left(1-v\,e^v\right)dx+\left(e^v-\frac{1}{v}\right)dy=0$$

$$\left(1-v\,e^v\right)dx+\left(e^v-\frac{1}{v}\right)\left(vdx+x\,dv\right)=0$$

$$\left(1-v\,e^v+v\,e^v-1\right)dx+\left(\frac{v\,e^v-1}{v}\right)x\,dv=0$$

$$\left(\frac{v\,e^{v}-1}{v}\right)x\,dv = 0 \quad \rightarrow \quad dv = 0 \quad \rightarrow \quad v = C$$

$$v = C \quad \rightarrow \quad \frac{y}{x} = C$$

$$y = C\,x$$

DIFFERENTIAL EQUATIONS AS MATHEMATICAL MODELS

Example 6. A cup of coffee at temperature $T = 80\ ^{o}C$ was left at a large room which temperature is $T_a = 22\ ^{o}C$. The heat transfer coefficient α is 0.001 (well insulated). Determine the temperature of the coffee after 10 minutes (min). Assume that the cooling process obeys Newton's law of cooling,

$$\frac{dT}{dt} = -\alpha\left(T - T_a\right)$$

where T is the temperature of the coffee and T_a is the room temperature (ambient) both Kelvin, α is the heat transfer coefficient in 1/s.

Determine the temperature of the coffee after 10 minutes (min).

Solution

From the previous notes on **MATHEMATICAL MODELS OF Des we have**

$$T = \left(T_o - T_a\right)e^{-\alpha\, t} + T_a$$

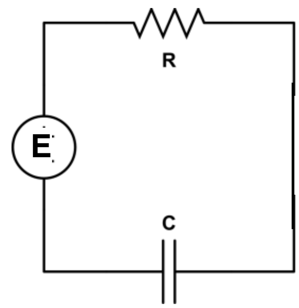
$$T = \left(\left[80\ ^{o}C + 273\right] - \left[22\ ^{o}C + 273\right]\right)e^{-0.001t} + \left[22\ ^{o}C + 273\right]$$

$$T = 58\,e^{-0.001t} + 295$$

At 10 min = 600 s

$$T = 58\,e^{-0.001(600)} + 295 = 326.8\ K = 53.8\ ^{o}C$$

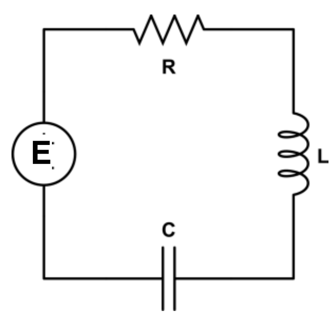
Example 7. A series circuit contains a resistor and a capacitor, as shown. Determine the differential equation that describes the charge q in the capacitor if the resistance is *R* and the capacitance is *C*, and the impressed voltage is *E (t)*.



Solution

Kirchhof's second law: ***the sum of voltage drop in each of the components is equal to the electrical voltage source,*** we have:

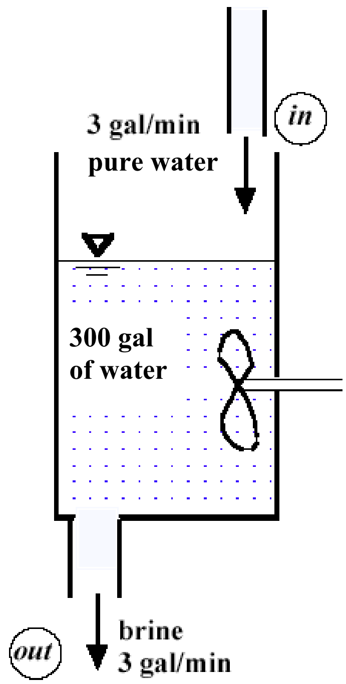
$$L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{1}{C}q=E(t)$$



For the circuit at hand the equation is,

$$R\frac{dq}{dt}+\frac{1}{C}q=E(t)$$

Example 8. Suppose that the large mixing tank initially holds 300 gallons of water in which 50 pounds of salt has been dissolved. Pure water is pumped into the tank at a rate of 3 gal/min, and when the solution is well stirred, it is pumped out at the same rate. Determine a differential equation for the amount $m_s(t)$ of salt in the tank at time t. What is $m_s(0)$?



Solution

$$\frac{d}{dt}m_s(t) = (\underset{\substack{\uparrow \\ 0\text{ (pure water)}}}{input\ rate\ of\ salt}) - (output\ rate\ of\ salt) = \underset{\substack{\uparrow \\ 0\text{ (pure water)}}}{R_{in}} - R_{out}$$

$$R_{out} = \frac{m_s(t)}{300}\, lb/gal\,(3\, gal/\min) = \frac{m_s(t)}{100}\, lb/\min$$

$$\frac{d\, m_s(t)}{dt} = -\frac{m_s(t)}{100}$$

$$\frac{d\, m_s(t)}{m_s(t)} m_s(t) = -\frac{1}{100}\, dt \quad \rightarrow \quad \ln(m_s(t)) = -\frac{t}{100} + C$$

$$m_s(t) = C' \exp\left(-\frac{t}{100}\right)$$

When $t = 0$, $m_s = 50\, lb$ $\rightarrow$ $C' = 50$ $\rightarrow$ $m_s(t) = 50 \exp\left(-\frac{t}{100}\right)$