Calculus and Vectors, MCV4U-B
Practice Test

Time: 2 hours
Total Marks: 100   Score: ____%

Instructions

The Practice Test uses the same format, type of questions, marking scheme, and length as the Final Test.

To get the most out of this Practice Test, allot yourself two uninterrupted hours, and don't use notes or books. You may use a scientific calculator.

Write your answers in the space provided.

The test has four (4) parts. An approximate time is given for each part. Look over the test before you begin, and leave some time to review your work at the end.

Part	Category	Marks	Time (min)
Preview			5
A	Knowledge and Understanding	46	30
B	Thinking	22	30
C	Communication	16	25
D	Application	16	25
Review			5
Total		100	120

1) rates of change
2) derivatives
3) curve sketching
4) exstensions
5) vectors

Part A: Knowledge and Understanding (46 marks) (*approximate time: 30 minutes*)

1. Evaluate.

a) $\lim_{x \to -1} \frac{x^2 + 9x - 4}{x^3}$ **(2 marks)**

$= \frac{(-1)^2 + 9(-1) - 4}{(-1)^3}$

$= \frac{-12}{-1}$

$= 12$

b) $\lim_{x \to -2} \frac{x^2 - 2x - 8}{x + 2}$ **(2 marks)**

$= \lim_{x \to -2} \frac{(x+2)(x-4)}{(x+2)}$

$= \lim_{x \to -2} (x-4)$

$= -2 - 4$

$= -6$

2. What is the average rate of change between $x = 1$ and $x = 3$ for the function $g(x) = -3(x-1)^2 + 11$? **(3 marks)**

question similar to 41 #4.

$g(x) = -3(x-1)^2 + 11$

$g(1) = -3((1)-1)^2 + 11 = 11$

$g(3) = -3((3)-1)^2 + 11 = -1$

$\frac{g(3) - g(1)}{3 - 1}$

$= \frac{-1 - 11}{3 - 1}$

ROC $= -6$

3. For the function $g(x) = 8x^2 - x + 4$, at what tangent point is the instantaneous rate of change equal to -1? **(3 marks)**

(1) derivative

$g'(x) = 16x - 1$

(2) set der = to -1 and solve for x

$16x - 1 = -1$

$x = 0$

(3) sub $x = 0$ back into og fxn

$g(0) = 8(0)^2 - (0) + 4$

$g(0) = 4$

$\therefore$, tangent point is $(0,4)$.

4. Determine the derivative.

quotient rule:
if $F(x) = \frac{f(x)}{g(x)}$
then, $F'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$

a) $y = \frac{x^3 - 3x + 4}{x + 3}$ **(2 marks)**

$f(x) = x^3 - 3x + 4$ $\quad g(x) = x + 3$
$f'(x) = 3x^2 - 3$ $\quad g'(x) = 1$

$y' = \frac{(3x^2-3)(x+3) - (x^3-3x+4)(1)}{(x+3)^2}$

$= \frac{3x^3 + 9x^2 - 3x - 9 - x^3 + 3x - 4}{(x+3)^2}$

$y' = \frac{2x^3 + 9x - 13}{(x+3)^2}$

b) $y = \ln(2x^3 + 3)$ **(2 marks)**

$y' = \frac{6x^2}{2x^3 + 3}$

c) $y = (3x^2 - 7x + 5)^3$ **(2 marks)**

$F'(x) = n(u)^{n-1}(u')$

$y' = 3(3x^2 - 7x + 5)^2(6x - 7)$

chain rule:
if $f(x) = u^n$
then $F'(x) = n(u)^{n-1}(u')$

d) $y = \sin^5 x$ **(2 marks)**

$y' = 5\sin^4 x \cos x$

5. Determine if the function $g(x) = 3x^3 - 2x$ is even, odd or neither? **(2 marks)**

$g(-x) = 3(-x)^3 - 2(-x)$
$= -3x^3 + 2x$
$g(-x) = -(3x^3 - 2x)$
$g(-x) = -g(x)$
$\therefore$ ODD.

symmetry:
even: $f(-x) = f(x)$
odd: $f(-x) = -f(x)$
neither: $f(-x) \neq f(x) \neq -f(x)$

6. Find the magnitude of the resultant of two vectors $\vec{u}$ and $\vec{v}$, given $|\vec{u}| = 5$, $|\vec{v}| = 4$ and the angle between the vectors when they are arranged tail to tail is 65º. **(2 marks)**

magnitude formula

$|\vec{r}| = \sqrt{|\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}||\vec{v}|\cos(\ )}$

$= \sqrt{(5)^2 + (4)^2 - 2(5)(4)\cos 115°}$ $\leftarrow 180° - 65° = 115°$

$= \sqrt{57.9}$

$\doteq 7.6$

7. Given $\vec{u} = [7,6,-1]$ and $\vec{v} = [2,3,4]$, determine the value of $|2\vec{u} + \vec{v}|$. **(2 marks)**

$2\vec{u} + \vec{v} = 2[7,6,-1] + [2,3,4] = [16,15,2]$

$|2\vec{u} + \vec{v}| = \sqrt{(16)^2 + (15)^2 + (2)^2}$
$= \sqrt{485}$
$\doteq 22.0$

adding vectors
if $\vec{u} = [u_1, u_2]$ and $\vec{v} = [v_1, v_2]$
$\vec{u} + \vec{v} = [u_1, u_2] + [v_1, v_2]$
$= [u_1 + v_1, u_2 + v_2]$
multiply by a scalar
let $\vec{u} = [u_1, u_2]$ and $k \in R$
$k\vec{u} = [ku_1, ku_2]$

$\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos\theta$

8. Determine the dot product of $\vec{u}=[2,5,1]$ and $\vec{v}=[-5,4,2]$. **(2 marks)**

$\vec{u}\cdot\vec{v}=(2)(-5)+(5)(4)+(1)(2)$
$=12$

dot product of cartesian
given $\vec{a}=[a_1,a_2]$ & $\vec{b}=[b_1,b_2]$
$\vec{a}\cdot\vec{b}=a_1b_1+a_2b_2$

9. Given vectors $\vec{u}=[2,4,-3]$ and $\vec{v}=[6,-2,-1]$, determine $\vec{u}\times\vec{v}$. **(2 marks)**

$\vec{u}\times\vec{v}=[((4)(-1)-(-3)(-2)),((-3)(6)-(2)(-1)),((2)(-2)-(4)(6))]$
$=[-10,-16,-28]$

cartesian cross product:
given $\vec{a}=[a_1,a_2,a_3]$ $\vec{b}=[b_1,b_2,b_3]$
$\vec{a}\times\vec{b}=[a_2b_3-a_3b_2, a_3b_1-a_1b_3, a_1b_2-a_2b_1]$

$a_1 \quad a_2 \quad a_3 \quad a_1 \quad a_2 \quad a_3$
$b_1 \quad b_2 \quad b_3 \quad b_1 \quad b_2 \quad b_3$

10. Given two vectors $\vec{u}=[14,-6]$ and $\vec{v}=[-2,5]$, determine the projection of $\vec{u}$ on $\vec{v}$. **(4 marks)**

we do not know angle.
$proj_{\vec{v}}\vec{u}=\left(\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v}$

find $\vec{u}\cdot\vec{v}$
$\vec{u}\cdot\vec{v}=(14)(-2)+(-6)(5)$
$\vec{u}\cdot\vec{v}=-58$
find $\vec{v}\cdot\vec{v}$
$\vec{v}\cdot\vec{v}=(-2)(-2)+(5)(5)$
$\vec{v}\cdot\vec{v}=29$

$proj_{\vec{v}}\vec{u}=\left(\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v}$
$=\left(\frac{-58}{29}\right)[-2,5]$
$=-2[-2,5]$
$proj_{\vec{v}}\vec{u}=[4,-10]$

projection: ↙ more RULES.

no angle
proj. of $\vec{u}$ on $\vec{v}$
$proj_{\vec{v}}\vec{u}=\left(\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v}$
proj of $\vec{v}$ on $\vec{u}$
$proj_{\vec{u}}\vec{v}=\left(\frac{\vec{v}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\right)\vec{u}$
or
$proj_{\vec{u}}\vec{v}=|\vec{v}|\cos\theta\left(\frac{1}{|\vec{u}|}\vec{u}\right)$

angle
if angle b/w $\vec{v}$ and $\vec{u}$ less than 90°
$|proj_{\vec{u}}\vec{v}|=|\vec{v}|\cos\theta$
or
$|proj_{\vec{u}}\vec{v}|=\frac{\vec{u}\cdot\vec{v}}{|\vec{u}|}$
b/w 90° and 180°
$|proj_{\vec{u}}\vec{v}|=-|\vec{v}|\cos\theta$
or
$|proj_{\vec{u}}\vec{v}|=-\frac{\vec{u}\cdot\vec{v}}{|\vec{u}|}$
angle = 90°
$proj_{\vec{u}}\vec{v}=0$

11. Find the equation of the tangent line to $f(x)=2x^3-4x+7$ at $(2,15)$. **(3 marks)**

tangent point (x_1,y_1) is point $(2,15)$
(1) derivative:
$f(x)=2x^3-4x+7$
$f'(x)=6x^2-4$
(2) the slope, m, is value of der. when $x=x_1=2$
$m=f'(2)$
$m=6(2)^2-4$
$=24-4$
$m=20$
(3) equation of line, in $y=m(x-x_1)+y_1$
$y=m(x-x_1)+y_1$
$y=20(x-2)+15$
$y=20x-40+15$
$y=20x-25$

equation of line:
$y=m(x-x_1)+y_1$

Similar questions to 2.2 in notes!

12. Given the vector equation $[x,y]=[3,-2]+t[8,7]$ find

a) the parametric equations **(1 mark)**

$x=3+8t$
$y=-2+7t$

2-space:
vector form:
$[x,y]=[x_0,y_0]+t[m_1,m_2]$

parametric form:
$x=x_0+tm_1$
$y=y_0+tm_2$

b) the symmetric equation **(1 mark)**

$\frac{x-3}{8}$ $\frac{y+2}{7}$

symmetric form:

$\frac{x-x_0}{m_1} = \frac{y-y_0}{m_2}$

c) the scalar equation **(2 marks)**

$\frac{x-3}{8} = \frac{y+2}{7}$

$7(x-3) = 8(y+2)$

$7x - 21 = 8y + 16$

$7x - 8y - 37 = 0$

scalar form:

$Ax + By + C = 0$

13. Find the intersection of $[x, y, z] = [1,3,-1] + t[1,1,4]$ and $[x, y, z] = [2,4,3] + s[3,1,-1]$. **(7 marks)**

Parametric:

$[x,y,z] = [1,3,-1] + t[1,1,4]$

$x = 1 + t$

$y = 3 + t$

$z = -1 + 4t$

$[x,y,z] = [2,4,3] + t[3,1,-1]$

$x = 2 + 3s$

$y = 4 + s$

$z = 3 - s$

Equate x's:

set both sets of parametric eqs equal and simplify

$1 + t = 2 + 3s$

$t - 3s = 1 \ (1)$

Equate y's:

$3 + t = 4 + s$

$t - s = 1 \ (2)$

Equate z's:

$-1 + 4t = 3 - s$

$4t + s = 4 \ (3)$

use elimination to solve equations from x and y:

$t - 3s = 1 \quad (1)$

$-(t - s = 1) \quad (2)$

$-2s = 0$

$s = 0$

find t:

$t - 3s = 1$

$t - 3(0) = 1$

$t = 1$

check in equation from z:

$4t + s = 4$

$4(1) + (0) = 4$

3-SP

vector form:

$[x, y, z] = [x_0, y_0, z_0] + t[a_1, a_2, a_3] + s[b_1, b_2, b_3]$

parametric form:

$x = x_0 + ta_1 + sb_1$

$y = x_0 + ta_2 + sb_2$

$z = x_0 + ta_3 + sb_3$

scalar form:

$Ax + By + Cz + D = 0$

where $[A,B,C]$ is a normal of the plane.

Substitute $t=1$ and $s=0$ into either set of parametric

$x = 1 + t = 1 + (1) = 2$

$y = 3 + t = 3 + (1) = 4$

$z = -1 + 4t = -1 + 4(1) = 3$

$\therefore$ point of intersection: $(2,4,3)$

Part B: Thinking (22 marks) (approximate time: 30 minutes)

14. For the function $f(x) = \dfrac{2x^2}{x^2 - 4}$, find the following

a) The x and y- intercepts. **(2 marks)**

x-int:
$0 = \dfrac{2x^2}{x^2-4}$
$0 = 2x^2$
$x = 0.$

y-int:
$y = \dfrac{2(0)^2}{(0)^2-4}$
$= \dfrac{0}{-4}$
$y = 0$

$\therefore$, x- and y-int are both $(0,0)$.

x-intercept:
Set $y=0$ and solve for x

y-intercept:
set $x=0$ and solve for y

b) The horizontal and vertical asymptotes. **(2 marks)**

vertical asymptote:
- restrictions from the denom.

$x^2 - 4 \neq 0$
$x^2 \neq 4$
$x \neq \pm 2$

$\therefore$, VA are $x=-2$ and $x=2$

horizontal asymptote:
- both numb/denom degree is 2.

$y = \dfrac{2}{1} = 2$

$\therefore$, HA is $y=2$

vertical asymptote:
comes from the value(s) of x that are not defined in the domain.

horizontal asymptote:
- if both numb. and denom. have same degree divide coefficents
- if numb degree less than denom then $y=0$.

c) The first and second derivatives. **(6 marks)**

quotient rule:
if $F(x) = \dfrac{f(x)}{g(x)}$
then, $F'(x) = \dfrac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$

$f(x) = 2x^2 \qquad g(x) = x^2 - 4$
$f'(x) = 4x \qquad g'(x) = 2x$

$$F'(x) = \frac{(4x)(x^2-4) - (2x^2)(2x)}{(x^2-4)^2}$$
$$= \frac{2x(2x^2 - 8 - 2x^2)}{(x^2-4)^2}$$
$$F'(x) = \frac{-16x}{(x^2-4)^2}$$

$f(x) = -16x \qquad g(x) = (x^2-4)^2$
$f'(x) = -16 \qquad g'(x) = 2(x^2-4)(2x)$

chain rule:
if $f(x) = u^n$
then $F'(x) = n(u)^{n-1}(u')$

$$f''(x) = \frac{-16(x^2-4)^2 - (-16x)(2(x^2-4)(2x))}{((x^2-4)^2)^2}$$
$$= \frac{-16(x^2-4)(x^2-4-4x^2)}{(x^2-4)^4}$$
$$= \frac{-16(x^2-4)(-3x^2-4)}{(x^2-4)^4}$$
$$= \frac{-16(-3x^2-4)}{(x^2-4)^3}$$

d) The intervals of increasing and decreasing **(3 marks)**

the critical numbers are $0, \pm 2$
$x=0$ b/c denom can not $= 0$, restriction.

intervals	$-16x$	$(x^2-4)^2$	$\frac{-16x}{(x^2-4)^2}$	in/decreasing
$x<-2$	+	+	+	increasing
$-2<x<0$	+	+	+	increasing
$0<x<2$	−	+	−	decreasing
$x>2$	−	+	−	decreasing

increasing intervals: $x<-2$ and $-2<x<0$.
decreasing intervals: $0<x<2$ and $x>2$

• $1^{st} = 0$
• table
• intervals.

e) Any local maximum or minimum points **(2 marks)**

the function changes increasing to decreasing at $x=0$.
∴, maximum point at $x=0$
maximum pt: $(0,0)$.
└ you plug x into original fxn

intervals of increasing/decreasing:
- decreasing to increasing: minimum
- increasing to decreasing: maximum.

f) The intervals of concavity **(3 marks)**

$f''(x) = \frac{-16(-3x^2-4)}{(x^2-4)^3}$

$0 = \frac{-16(-3x^2-4)}{(x^2-4)^3}$

$0 = -16(-3x^2-4)$

$0 = -3x^2-4$

$0 =$ n/a

vertical asymptote/restrictions are $x=\pm 2$.

intervals	-16	$(-3x^2-4)$	$(x^2-4)^3$	$\frac{-16(-3x^2-4)}{(x^2-4)^3}$	concavity
$x<-2$	−	−	+	+	up
$-2<x<2$	−	−	−	−	down
$x>2$	−	−	+	+	up

concave up: $x<-2$ and $x>2$
concave down: $-2<x<2$

intervals of concavity:
- find 2^{nd} derivative
- find critical #'s
- intervals
- table
- concave up/down intervals

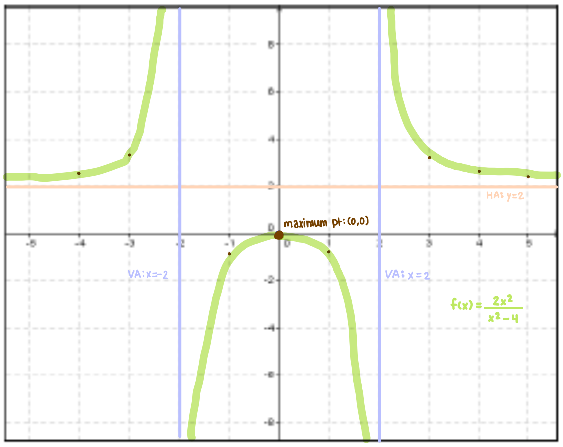
g) Any inflection points **(2 marks)**

inflection point:
Function changes from concave up↗ to down↘, could be a a POI at $x=-2$.
however, $x=-2$ is a restriction on the domain; NO POI.

Function changes from concave down↘ to up↗, could be a a POI at $x=2$
however, $x=2$ is a restriction on the domain; NO POI.

infection point
- concave up to down:
concave down to up:

h) Now sketch a graph of the function. **(2 marks)**



Part C: Communication (16 marks) *(approximate time: 25 minutes)*

15. Define the terms "concavity" and "inflection point" and describe how you would find the intervals of concavity on a graph given the equation of a function. **(3 marks)**

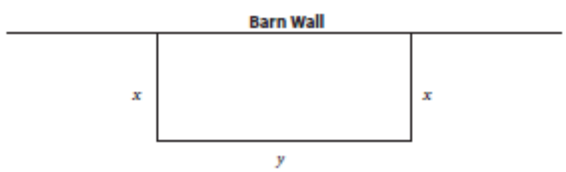
Concavity:
- curvature of a graph. Two kinds of concavity up and down.
- concave up: parabola that opens upwards Concave down: downwards

Inflection point:
- point where a graph

MAXIMIZE AREA

16. Your friend is having difficulty answering the following question. List the steps both algebraically and in words to show your friend how to find the solution for the question. **(7 marks)**

A farmer wishes to enclose a rectangular area adjacent to one of his barns with 280 m of fencing. Three sides will be fenced, and the barn will make up the fourth side. Determine the dimensions of the maximum area that can be produced with this amount of fencing.



Optimization
(1) Establish variables and equation
- Let statement
- Diagram
- Establish equation

(2) Find the derivative
(3) set der = 0 and find critical number(s)
(4) find second der = 0 and confirm max/min.
(5) conclusion.

identify variables
Let x rep. the width of the area in m.
Let y rep. the length of the area in m.
Let A rep. the area in m².
Let P rep. the perimeter of the area in m.

Function for area
$A = xy$

constraint
$P = 2x + y$
$280 = 2x + y$
$y = 280 - 2x$

Substitute the constraint into the function

$A = xy$

$A = x(280 - 2x)$

$A = 280x - 2x^2$

Find derivative

$A'(x) = 280 - 4x$

set der $= 0$ and solve for x.

$0 = 280 - 4x$

$4x = 280$

$x = 70$

Solve for y.

$y = 280 - x$

$y = 280 - (70)$

$y = 140$

The dimension to maximize the area are 70m by 140m.

17. Show that the curve $y = 2x^3 + 3x - 4$ has no tangents with slope 2. **(2 marks)**

$y = 2x^3 + 3x - 4$

$m = 2$

derivative:

$y' = 6x^2 + 3$

$2 = 6x^2 + 3$

$x^2 = -\frac{1}{6}$

$x = \pm\sqrt{\frac{1}{6}}$

No sol. for x. $\therefore$ no possible tangents w/ slope 2.

18. If $\vec{u} = [6,1,8]$ is orthogonal to $\vec{v} = [6,-4,k]$, determine the value(s) of k. **(2 marks)**

orthogonal: perpendicular vectors, dot product is 0.

$\vec{u} \cdot \vec{v} = (6)(6) + (1)(-4) + (8)(k)$

$0 = 36 - 4 + 8k$

$0 = 32 + 8k$

$8k = -32$

$k = -4$.

dot product 3-space

given $\vec{u} = [u_1, u_2, u_3]$ & $\vec{v} = [v_1, v_2, v_3]$

$\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_3v_3$

19. Find the derivative of $x^2 - xy = 14$ using implicit differentiation. **(2 marks)**

$x^2 - xy = 14$

$2x - ((1)(y) + (x)(1y')) = 0$

$2x - y - xy' = 0$

$xy' = 2x - y$

$y' = \frac{2x - y}{x}$

Part D: Application (16 marks) *(approximate time: 25 minutes)*

20. A farmer has 400 bushels of apples that he can sell at \$24 each today. For each week he waits the price decreases by \$2 and the number of bushels increases by 100. For each additional week his storage cost is \$600. How many weeks should he wait to have

a) maximum revenue? **(3 marks)**

Let x rep. the number of weeks the farmer should wait.
Let R rep. the revenue in dollars.

$R = (24-2x)(400+100x)$
$R = -200x^2 + 1600x + 9600$

maximize:
$R' = -400$
$0 = -400x + 1600$
$400x = 1600$
$x = 4$
the farmer should wait 4 weeks to maximize the revenue.

b) maximum profit? **(4 marks)**

Let x rep. the number of weeks the farmer should wait.
Let P rep. the profit in dollars.

$P = (24-2x)(400+100x - 600x$
$= -200x^2 + 1600x + 9600 - 600x$
$P = -200x^2 + 1000x + 9600$

maximize:
$P' = -400x + 1000$
$0 = -400x + 1000$
$400x = 1000$
$x = 2.5$
the farmer should wait 2.5 weeks to maximize the profit.

21. A ladder 25 m long is leaning against a vertical wall. If the bottom of the ladder is pulled horizontally away from the wall at 3 m/s, how fast is the top of the ladder sliding down the wall, when the bottom is 15 m from the wall? **(4 marks)**

variables:
Let x rep. the horizontal distance from the wall in m.
Let y rep. the vertical distance up the wall in m.

given:
$\frac{dx}{dt} = 3$

find:
$\frac{dy}{dt}$

when:
$x = 15$

$25^2 = x^2 + y^2$

$0 = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$

use pythagorean theorem to find $y = 20$ when $x = 15$

$25^2 = x^2 + y^2$

$0 = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$

$0 = 2(15)(3) + 2(20)\left(\frac{dy}{dt}\right)$

$0 = 90 + 40\frac{dy}{dt}$

$-90 = 40\frac{dy}{dt}$

$\frac{dy}{dt} = \frac{-90}{40}$

$\frac{dy}{dt} = -2.25$

$\therefore$, the ladder sliding down the wall at 2.25m/s.

related rates

(1) Given
- diagram
- define all variables
- rates
- state values given

(2) Find
- variable / rate you want

(3) when
- condition that clarifies the inst ROC working w/

(4) Solve
- find an equation or formula that relates the variables
- find derivative
- substitute conditional value into derivative

(5) statement
- $\therefore$, conclusion.

22. A plane is travelling at 540 km/h on a bearing of 45° and a wind of 55 km/h is blowing from a bearing of 270°. Find the plane's speed and bearing. **(5 marks)**

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(1) variables

Let x rep the angle's speed in km/h

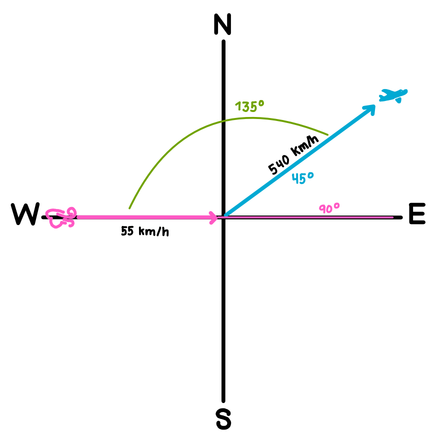
Let θ rep the angle b/w the resultant and the plane path in degree.

(2) angle

bearing angle 45° and the bearing angle of wind is from 270°

angle across from the resultant is 135°

$90+45°$



resultant: use cosine law.

$|\vec{x}|^2 = (540)^2 + (55)^2 - 2(540)(55)\cos 135$

$|\vec{x}|^2 = 336627.1$

$|\vec{x}| = 580.2$

resulant vector has a length of 580.2 km/h.

∴, the ground speed of the aircraft is 580.2 km/h.

cosine law:

$a^2 = b^2 + c^2 - 2bc \times \cos\theta$

bearing: use sine law

resultant → $\frac{580.2}{\sin 135} = \frac{540}{\sin\theta}$ ← bearing km/h, knots

$540 \sin 135 = 580.2 \sin\theta$

$\sin\theta = \frac{540 \sin 135}{580.2}$

$\theta = 41.2°$

(ends in quadrant 1 ∴ bearing is 41.2°.)

sine law:

resultant → $\frac{a}{\sin a} = \frac{b}{\sin\theta}$ ← bearing km/h, knots

angle in cosine law.