SYSC-3200 Winter 2021: Assignment 4 Solutions

Question 1 [10 marks]

[formulation 4 marks] [stages 5 marks] [summary statement 1 mark]

Formulation:

- *Stages*: the crop under consideration
- *State at a stage*: how much area is left on the farm?
- *Decision at a stage*: how many fields of crop *t* to plant?
- *Decision update to state*: the area remaining in the farm is reduced appropriately
- Recursive value relationship
  - x_t: # of fields of crop *t*
  - r_t: profit of crop *t* per field
  - w_t: area required by a single field of crop *t*
  - d_t: area left in farm when considering crop *t*
  - $f_t(d_t)$: maximum profit obtainable considering the framer enters stage *t* with a remaining area of d_t
  - $f_t(d_t) = \max_{x_t} \{r_t x_t + f_{t+1}(d_t - w_t x_t)\}$ where $0 \le x_t \le d_t / w_t$ and integer

<u>At stage 4</u>

$f_4(d_4) = \max_{x_4} \{2700x_4\}$

d_4	$x_4 = 0$	$x_4 = 1$	$x_4 = 2$	$f_4(d_4)$
0-6	**<u>0</u>**	-	-	0
7-13	0	**<u>2700</u>**	-	2700
14-15	0	2700	**<u>5400</u>**	5400

<u>At stage 3</u>

$f_3(d_3) = \max_{x_3} \{1800x_3 + f_4(d_3 - 6\, x_3)\}$

d_3	$x_3 = 0$	$x_3 = 1$	$x_3 = 2$	$f_3(d_3)$	d_4
0-5	**<u>0</u>**	-	-	0	0-5
6	0	**<u>1800</u>**	-	1800	0
7-11	**<u>2700</u>**	1800	-	2700	7-11
12	2700	1800	**<u>3600</u>**	3600	0
13	2700	**<u>4500</u>**	3600	4500	7
14	**<u>5400</u>**	4500	3600	5400	14
15	**<u>5400</u>**	4500	3600	5400	15

<u>At stage 2</u>

$f_2(d_2) = \max_{x_2} \{1600x_2 + f_3(d_2 - 3x_2)\}$, where $0 \le x_2 \le 4$

d_2	$x_2 = 0$	$x_2 = 1$	$x_2 = 2$	$x_2 = 3$	$x_2 = 4$	$f_2(d_2)$	d_3
0	**0**	-	-	-	-	0	0
1	**0**	-	-	-	-	0	1
2	**0**	-	-	-	-	0	2
3	0	**1600**	-	-	-	1600	0
4	0	**1600**	-	-	-	1600	1
5	0	**1600**	-	-	-	1600	2
6	1800	1600	**3200**	-	-	3200	0
7	2700	1600	**3200**	-	-	3200	1
8	2700	1600	**3200**	-	-	3200	2
9	2700	3400	3200	**4800**	-	4800	0
10	2700	4300	3200	**4800**	-	4800	1
11	2700	4300	3200	**4800**	-	4800	2
12	3600	4300	5000	4800	**6400**	6400	0
13	4500	4300	5900	4800	**6400**	6400	1
14	5400	4300	5900	4800	**6400**	6400	2
15	5400	5200	5900	**6600**	6400	6600	6

At stage 1

$f_1(d_1) = \max_{x_1} \{1200x_1 + f_2(15 - 2x_1)\}$, where $0 \le x_1 \le 5$

d_1	$x_1 = 0$	$x_1 = 1$	$x_1 = 2$	$x_1 = 3$	$x_1 = 4$	$x_1 = 5$	$f_1(25)$	d_2
15	6600	7600	7200	**8400**	8000	7600	8400	9

To recover the solution:

- Stage 1: plant 3 soy fields
- Stage 2: plant 3 wheat fields
- Stage 3: plant 0 canola fields
- Stage 4: plant 0 corn fields

So the farmer will have a maximum profit of $8,400,000 by planting 3 soy fields, 3 wheat fields, 0 canola fields and 0 corn fields.

Question 2 [10 marks]

a) [6 marks] The distribution of the loons at the end of hour 1 is

$$\begin{vmatrix} 0.25 & 0.45 & 0.30 \end{vmatrix} \times \begin{vmatrix} 0.1 & 0.3 & 0.6 \\ 0.2 & 0.3 & 0.5 \\ 0.3 & 0.3 & 0.4 \end{vmatrix} = \begin{vmatrix} 0.2050 & 0.3000 & 0.4950 \end{vmatrix}$$

The distribution of the loons at the end of hour 2 is

$$\begin{vmatrix} 0.2050 & 0.3000 & 0.4950 \end{vmatrix} \times \begin{vmatrix} 0.1 & 0.3 & 0.6 \\ 0.2 & 0.3 & 0.5 \\ 0.3 & 0.3 & 0.4 \end{vmatrix} = \begin{vmatrix} 0.2290 & 0.3000 & 0.4710 \end{vmatrix}$$

The distribution of the loons at the end of hour 3 is

$$\begin{vmatrix} 0.2290 & 0.3000 & 0.4710 \end{vmatrix} \times \begin{vmatrix} 0.1 & 0.3 & 0.6 \\ 0.2 & 0.3 & 0.5 \\ 0.3 & 0.3 & 0.4 \end{vmatrix} = \begin{vmatrix} 0.2242 & 0.3000 & 0.4758 \end{vmatrix}$$

A summary of the percentage of loons at each food source at the end of each hour is below:

Food source	**initial**	**after 1 hr**	**after 2 hrs**	**after 3 hrs**
A	0.25	0.2050	0.2290	0.2242
B	0.45	0.3000	0.3000	0.3000
C	0.3	0.4950	0.4710	0.4758

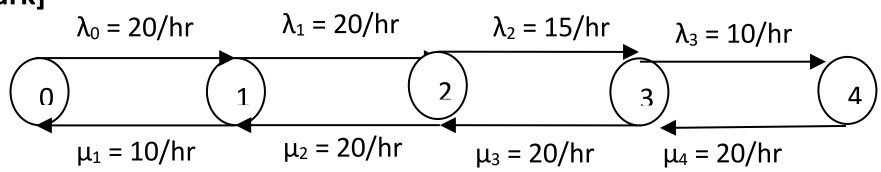
b) [4 marks] The steady state probability distribution is found by solving this system of linear equations:

Balance at food source A: $P_A = 0.1P_A + 0.3P_B + 0.6P_C$
Balance at food source B: $P_B = 0.2P_A + 0.3P_B + 0.5P_C$
Overall condition: $1 = P_A + P_B + P_C$

Yes, steady state will be reached eventually. The steady state solution is $P_A = 0.225$, P_B=0.3, and $P_C = 0.475$. Hence, at steady state 22.5% of the loons will be at food source A, 30% of the loons will be at food source B, and 47.5% of the loons will be at food source C.

Question 3 [10 marks]

a) [1 mark]



b) [3 marks] The probability of having 0 seniors in the system is:

$P_0 = \frac{1}{1+\sum_{n=1}^{\infty} C_n}$

$C_1 = \frac{\lambda_0}{\mu_1} = \frac{20}{10} = 2.0$ $\quad$ $C_2 = \frac{\lambda_1}{\mu_2} C_1 = \frac{20}{20}(2.0) = 2.0$

$C_3 = \frac{\lambda_2}{\mu_3} C_2 = \frac{15}{20}(2.0) = 1.5$ $\quad$ $C_4 = \frac{\lambda_3}{\mu_4} C_3 = \frac{10}{20}(1.5) = 0.75$

$P_0 = \frac{1}{1+[2.0+2.0+1.5+0.75]} = 0.137931$ or 13.7931%

The probability of having 1 seniors in the system is:
$P_1 = C_1 P_0 = (2.0)(0.137931) = 0.275862$ or 27.5862%

The probability of having 2 seniors in the system is:
$P_2 = C_2 P_0 = (2.0)(0.137931) = 0.275862$ or 27.5862%

The probability of having 3 seniors in the system is:
$P_3 = C_3 P_0 = (1.5)(0.137931) = 0.2068965$ or 20.68965%

The probability of having 4 seniors in the system is:
$P_4 = C_4 P_0 = (0.75)(0.137931) = 0.1034482$ or 10.34482%

c) [3 marks] The expected number of seniors in the system is:

$L = \sum_{n=0}^{\infty} n\, P_n = (0)(0.137931) + (1)(0.275862) + (2)(0.275862) + (3)(0.2068965) + (4)(0.1034482) = 1.8621$

d) [3 marks] The expected waiting time of seniors in the system is:

$\omega = \frac{L}{\bar{\lambda}}$

$\bar{\lambda} = \sum_{n=0}^{\infty} \lambda_n P_n = (20)(0.137931) + (20)(0.275862) + (15)(0.275862) + (10)(0.2068965) = 14.483/\text{hr}$

$\omega = \frac{1.8621}{14.483} = 0.12857$ hours, or about 7 minutes and 43 seconds.