MATH2004 - Review of Some Formulas

1. Fourier Series

Let $f(x)$ be $2L$-periodic function. Then $f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right]$

$$a_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}dx,\ b_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{m\pi x}{L}dx.$$

$$[f(x^+) + f(x^-)]/2 = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right].$$

Let $f(x)$ be a function defined on $[0, L]$. The *sine series of* $f(x)$ is the sine series of the $2L$-periodic odd extension of $f(x)$. The *cosine series of* $f(x)$ is the cosine series of the $2L$-periodic even extension of $f(x)$.

2. Parametric equations x=f(t), y=g(t).

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}, \qquad \frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}, \text{ provided } \frac{dx}{dt} \neq 0.$$

Area: $A = \int_a^b y\,dx = \int_\alpha^\beta y(t)x'_t\,dt$, where $x(\alpha) = a$, $x(\beta) = b$.

Arc Length: $L = \int_a^b \sqrt{1+(y'_x)^2}dx = \int_\alpha^\beta \sqrt{1+(y'_t/x'_t)^2}x'_t\,dt = \int_\alpha^\beta \sqrt{(x'_t)^2+(y'_t)^2}dt$,

where $x(\alpha) = a$, $x(\beta) = b$.

Area of a surface obtained by rotating a curve about x-axis: $S = \int_\alpha^\beta 2\pi y(t)\sqrt{x'^2_t + y'^2_t}\,dt$.

Polar coordinates $x = r\cos\theta$, $y = r\sin\theta$; $r = \sqrt{x^2+y^2}, \tan\theta = \frac{y}{x}$.

Tangents to Polar Curves: $y'_x = \frac{y'_\theta}{x'_\theta} = \frac{r'_\theta \sin\theta + r\cos\theta}{r'_\theta\cos\theta - r\sin\theta}$.

Area: $A = \int_\alpha^\beta \frac{1}{2}(\mathbf{r}_o^2 - \mathbf{r}_i^2)d\theta$.

Arc Length: $L = \int_\alpha^\beta \sqrt{x'^2_\theta + y'^2_\theta}\,d\theta = \int_\alpha^\beta \sqrt{r^2 + r'^2_\theta}\,d\theta$.

3. Vectors

Dot product and Cross product: Let $\mathbf{a}=\langle a_1, a_2, a_3\rangle$, $\mathbf{b}=\langle b_1, b_2, b_3\rangle$. Then

$\mathbf{a}\bullet\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$.

$$\mathbf{a}\times\mathbf{b} = \langle a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

Projection: $\text{proj}_{\mathbf{a}}\,\mathbf{b} = \left(\dfrac{\mathbf{ab}}{|\mathbf{a}|}\right)\dfrac{\mathbf{a}}{|\mathbf{a}|} = \dfrac{\mathbf{ab}}{|\mathbf{a}|^2}\mathbf{a}$, $\text{comp}_{\mathbf{a}}\,\mathbf{b} = \dfrac{\mathbf{ab}}{|\mathbf{a}|}$.

Equations of Lines through a point $P_0 = (x_0, y_0, z_0)$ with direction v = <a,b,c>:

(1) Vector equation: $\mathbf{r} = \mathbf{r}_0 + t\,\mathbf{v} = \langle x_0, y_0, z_0\rangle + t\langle a, b, c\rangle$,

(2) The parametric equation: $x = x_0 + at$, $y = y_0 + bt$, $z = z_0 + ct$.

(3) Symmetric equations: $\dfrac{x - x_0}{a} = \dfrac{y - y_0}{b} = \dfrac{z - z_0}{c}$.

Plane through a point $P_0 = (x_0, y_0, z_0)$ with normal vector v = <a,b,c>:

$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$.

The distance from a point $P_0 = (x_0, y_0, z_0)$ to a plane ax+by+cz+d=0: .

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

4. Vector function $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$

Integrals: $\int \mathbf{r}(t)dt = \left(\int f(t)dt, \int g(t)dt, \int h(t)dt\right)$.

Arc Length**:** $L = \int_a^b |\mathrm{r}'(t)|\,dt = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\,dt$.

Unit tangent vector : $\mathbf{T}(t) = \mathbf{r}'(t) / |\mathbf{r}'(t)|$.

The curvature of a curve $\mathbf{r}(t)$ is: $\kappa(t) = \left|\dfrac{d\mathbf{T}}{ds}\right| = \left|\dfrac{d\mathbf{T}/dt}{ds/dt}\right| = \left|\dfrac{\mathbf{T}'}{s'}\right| = \dfrac{|\mathbf{T}'|}{|\mathbf{r}'|} = |\mathbf{r}'\times\mathbf{r}''| / |\mathbf{r}'|^3$.

Principal unit normal vector $\mathbf{N} = \mathbf{T}' / |\mathbf{T}'|$

Binormal vector $\mathbf{B} = \mathbf{T}\times\mathbf{N}$

Velocity: $\mathbf{v} = \mathbf{r}'(t)$, and *speed* $= |\mathbf{r}'(t)|$.

Acceleration: $\mathbf{a} = \mathbf{r}''(t) = v'\,\mathbf{T} + \kappa v^2\mathbf{N}$.

The tangential component is $a_T = v' = \mathbf{r}'\cdot\mathbf{r}'' / |\mathbf{r}'|$ and the normal component is $a_N = \kappa v^2$.

5. Partial Derivatives

Differentiable: $df(x, y) = f_x\,dx + f_y\,dy$, $df(x, y, z) = f_x\,dx + f_y\,dy + f_z\,dz$.

The Chain Rule: Suppose $z = f(x, y)$.

Case 1. If $x = g(t)$, $y = h(t)$, then $z_t = z_x g_t + z_y h_t$

Case 2. If $x = g(s, t)$ and $y = h(s, t)$, then $z_s = z_x x_s + z_y y_s$, $z_t = z_x x_t + z_y y_t$.

The equation of the tangent line of a curve defined implicitly by $F(x, y) = c$ *is:*

$y = -(F_x(x_0, y_0) / F_y(x_0, y_0))(x - x_0) + y_0$.

The tangent plane of a surface $z = f(x, y)$ at a point $P(x_0, y_0, z_0)$ is

$f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) - (z - z_0) = 0$

The tangent plane of a surface defined implicitly by the equation $F(x, y, z) = c$ at a point (x_0, y_0, z_0) is:

$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$.

The directional derivative of $f(x, y)$ at (x_0, y_0) in the direction of a unit vector **u** is:

$D_{\mathbf{u}} f(x, y) = f_x(x, y)\,a + f_y(x, y)\,b$, $\mathbf{u} = \langle a, b\rangle$

$D_{\mathbf{u}} f(x, y, z) = f_x(x, y, z)\,a + f_y(x, y, z)\,b + f_z(x, y, z)\,c$, $\mathbf{u} = \langle a, b, c\rangle$.

Gradient vector:

grad $f(x, y)$ or $\nabla f(x, y) = \langle f_x(x, y), f_y(x, y)\rangle$.

grad $f(x, y, z)$ or $\nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z)\rangle$.

Critical points: $f_x = 0$, and $f_y = 0$, or one of f_x and f_y does not exits.

Second derivative test: Let (x_0, y_0) be a critical point. $D(x, y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx} f_{yy} - f_{xy}^2$.

(i) If $D(x_0, y_0) < 0$, then (x_0, y_0) is a saddle point.

(ii) If $D(x_0, y_0) > 0$, and $f_{xx}(x_0, y_0) > 0$, $f(x_0, y_0)$ is a local minimum.

(iii) If $D(x_0, y_0) > 0$, and $f_{xx}(x_0, y_0) < 0$, $f(x_0, y_0)$ is a local maximum.

Lagrange Multipliers**.** To find the extreme values of the function $w = f(x, y, z)$ subject to constraints $g(x, y, z) = 0$, $h(x, y, z) = 0$, solve the system of equations:

$\nabla f = \lambda \nabla g + \mu \nabla h$,

$g(x, y, z) = 0$,

$h(x, y, z) = 0$.

For each solution (x, y, z, λ, μ) of this system of equations, find the value of $w(x, y, z)$. The maximum is the maximum value of w, and the minimum is the minimum of w.

To find the extreme values of the function f subject to one constraint $g = 0$, solve the system of equation: $\nabla f = \lambda \nabla g$, $g = 0$,

6. Double Integral

Fubini's Theorem: $$\iint_S f(x,y)dA = \int_a^b \int_c^d f(x,y)dydx = \int_c^d \int_a^b f(x,y)dxdy$$

Type I: $D = \{(x, y); a \le x \le b, g_1(x) \le y \le g_2(x)\}$. Then

$$\iint_D f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx$$

Type II: $D = \{(x, y); c \le y \le d, h_1(y) \le x \le h_2(y)\}$. Then

$$\iint_D f(x,y)dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dxdy.$$

The double integral over a polar rectangle R is:

$$\iint_R f(x,y)dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)rdrd\theta\ , \ dA = rdrd\theta$$

7. Triple Integral

Fubini's theorem: $$\iiint_B f(x,y,z)dV = \int_r^s \int_c^d \int_a^b f(x,y,z)dxdydz\ .$$

Type I. $E = \{(x, y, z); (x, y) \in D, u_1(x, y) \le z \le u_2(x, y)\}$, where D is the region in (x, y) plane bounded by the graph of $F(x, y) = 0$. Then

$$\iiint_E f(x,y,z)dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)dz \right] dA\ .$$

Triple Integrals in Cylindrical Coordinate System: Suppose the region E of integration is of type I. Let D be the projection of E onto the xy-plane. Then

$$\iiint_E f(x,y,z)dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(r\cos\theta, r\sin\theta, z)dz \right] dA\ ,$$

where $x = r\cos\theta$, $y = r\sin\theta$, z = z, $dV = r\, dzdrd\theta$.

Triple Integrals in Spherical Coordinate System: If a region E is specified in spherical coordinates, then

$$\iiint_E f(x,y,z)dV = \iiint_E f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2 \sin\phi d\rho d\theta d\phi\ .$$

where $x = \rho \sin \phi \cos \theta$, $y = \rho \sin \phi \sin \theta$, $z = \rho \cos \phi$, $dV = \rho^2 \sin \phi \ d\rho\, d\theta\, d\phi$.

8. Vector Calculus

Conservative Vector Fields: F = ∇f, where f is called the potential function of F.

Line Integrals of Scalar fields

$$\int_C f(x,y)ds = \int_a^b f[x(t), y(t)]\sqrt{x'^2_t + y'^2_t}\,dt.$$

$$\int_C f(x,y,z)ds = \int_a^b f[x(t), y(t), z(t)]\sqrt{x'^2_t + y'^2_t + z'^2_t}\,dt.$$

Line Integral of Vector fields:

$$\int_C \mathbf{F} \bullet d\,\mathbf{r} = \int_C P(x,y)dx + Q(x,y)dy,$$

$$\int_C \mathbf{F} \bullet d\,\mathbf{r} =$$

$$\int_C P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)$$

$$= \int_a^b [P(u(t),v(t))u'(t) + Q(u(t),v(t))v'(t) + R(u(t),v(t),w(t))w'(t)]dt.$$

Fundamental Theorem for Line Integral: Suppose $\mathbf{F} = \nabla f$ and C is a smooth curve from a to b. Then $\int_C \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$.

A Vector Field $\mathbf{F} = \langle P, Q \rangle$ is Conservative iff $P_y = Q_x$