MATH2004A – Test 3 – 1:35 pm - 2:25 pm, Nov. 14
Name: **Student Number:**
Total: 15 marks (for 4 questions). You may write on **both sides**.

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1. [4 points] Consider the space curve C: $r(t) =< \cos t, \sin(2t), t+1 >$. (i) Find the tangent line at $t = 0$. (ii) Find the curvature at $t = 0$.

Solution:
(i) $r(0) =< 1, 0, 1 >$. $r'(t) =< -\sin t, 2\cos 2t, 1 >$.
$r'(0) =< 0, 2, 1 >$ [0.5 point].
The tangent line is [1 point]

$$r =< 1, 0, 1 > +t < 0, 2, 1 > .$$

(ii) $r''(t) =< -\cos t, -4\sin 2t, 0 >$.
$r''(0) =< -1, 0, 0 >$ [0.5].
The curvature is [1 point for the first equality; 1 point for evaluation]

$$\kappa = \frac{|r'(0) \times r''(0)|}{|r'(0)|^3} = \sqrt{5}/(\sqrt{5})^3 = 1/5.$$

2. [4 points] (i) Find the tangent plane of the surface $2x^2 + y^2 - xz^3 = 2$ at the point $(1, 1, 1)$. (ii) Use the second derivative test to classify the point $(0, 0)$ for $f(x, y) = x^2 - 3xy + y^2$.

solution:
(i) Denote $F(x, y, z) = 2x^2 + y^2 - xz^3$.
Then

$$\bigtriangledown F =< 4x - z^3, 2y, -3xz^2 > .$$

So [1 point]

$$\bigtriangledown F(1, 1, 1) =< 3, 2, -3 > .$$

The tangent plane is [1 point]

$$3(x - 1) + 2(y - 1) - 3(z - 1) = 0.$$

(ii) First, $(0, 0)$ is a critical point. We have

$$f_{xx} = 2, \quad f_{xy} = -3$$
$$f_{yx} = -3, \quad f_{yy} = 2.$$

$f_{xx} > 0$, and the determinant is

$$D = \begin{vmatrix} 2 & -3 \\ -3 & 2 \end{vmatrix} < 0.[1 point]$$

So $(0, 0)$ is a saddle point [1 point].

3. [3 points] Suppose $z = f(x, y) = (2x + y)^3 + x^2$, $x = t + s$, $y = t - s$. Evaluate $\partial z/\partial s$ at $t = 1$ and $s = 1$.

Solution: We have [1 point]

$$z_x = 3(2x + y)^2(2) + 2x, \quad z_y = 3(2x + y)^2.$$

When $t = 1$, $s = 1$, we have $x = 2$ and $y = 0$, and

$$z_x = 96 + 4 = 100, \quad z_y = 48.$$

Next [1 [point]

$$x_s = 1, \quad y_s = -1.$$

Finally [1 point]

$$\partial z/\partial s = z_x x_s + z_y y_s = (100)(1) + (48)(-1) = 52.$$

4. [4 points] Use the Lagrange multiplier to find the extreme values of $f(x, y) = x - 2y$ subject to the constraint $x^2 + 2y^2 = 1$.

Solution: Denote $F(x, y) = x - 2y - \lambda(x^2 + 2y^2 - 1)$.
We write the equations [2 points]

$$F_x = 1 - 2\lambda x = 0, \tag{1}$$
$$F_y = -2 - 4\lambda y = 0, \tag{2}$$
$$x^2 + 2y^2 = 1. \tag{3}$$

We have $1/(4\lambda^2) + 2/(4\lambda^2) = 1$. The two sets of solutions are [1 point]

$$(\lambda = \sqrt{3}/2, x = 1/\sqrt{3}, y = -1/\sqrt{3}), \quad (\lambda = -\sqrt{3}/2, x = -1/\sqrt{3}, y = 1/\sqrt{3}).$$

We have [1 point]

$$f(1/\sqrt{3}, -1/\sqrt{3}) = \sqrt{3}, \text{(maximum)}$$

and

$$f(-1/\sqrt{3}, 1/\sqrt{3}) = -\sqrt{3}, \text{(minimum)}$$