PHY2361 Assignment 8 solutions 2021

A. a) In class notes (+ in textbook), considered an infinite square well with $V(x') = \begin{cases} 0 & \text{for } 0 \leq x' \leq L' \\ \infty & \text{otherwise} \end{cases}$ which in the t-dependent Schrodinger equation gave wave function solutions

$\psi_n(x') = \sqrt{\frac{2}{L'}} \sin\left(\frac{n\pi x'}{L'}\right)$ inside the well.

By letting $L' = 2L/3$ and $x' = x + L/3$ we get

$$V(x) = \begin{cases} 0 & \text{for } 0 \leq x + L/3 \leq \frac{2L}{3} \text{ i.e. for } -L/3 \leq x \leq L/3 \\ \infty & \text{otherwise} \end{cases}$$

as required here.

$$\text{and } \psi_n(x) = \sqrt{\frac{2}{2L/3}} \sin\left(\frac{n\pi(x + L/3)}{2L/3}\right) = \sqrt{\frac{3}{L}} \sin\left(\frac{3n\pi}{2L}x + \frac{n\pi}{2}\right)$$

for $-L/3 \leq x \leq L/3$

and $\psi_n(x) = 0$ for $|x| > L/3$.

$$\therefore \psi_1(x) = \sqrt{\frac{3}{L}} \sin\left(\frac{3\pi x}{2L} + \frac{\pi}{2}\right) = \sqrt{\frac{3}{L}} \cos\left(\frac{3\pi x}{2L}\right)$$

and $|\psi_1(x)|^2 = \frac{3}{L}\cos^2\left(\frac{3\pi x}{2L}\right)$ is the probability density in state $n=1$

$$\psi_2(x) = \sqrt{\frac{3}{L}} \sin\left(\frac{6\pi x}{2L} + \pi\right) = (-)\sqrt{\frac{3}{L}} \sin\left(\frac{3\pi x}{L}\right)$$

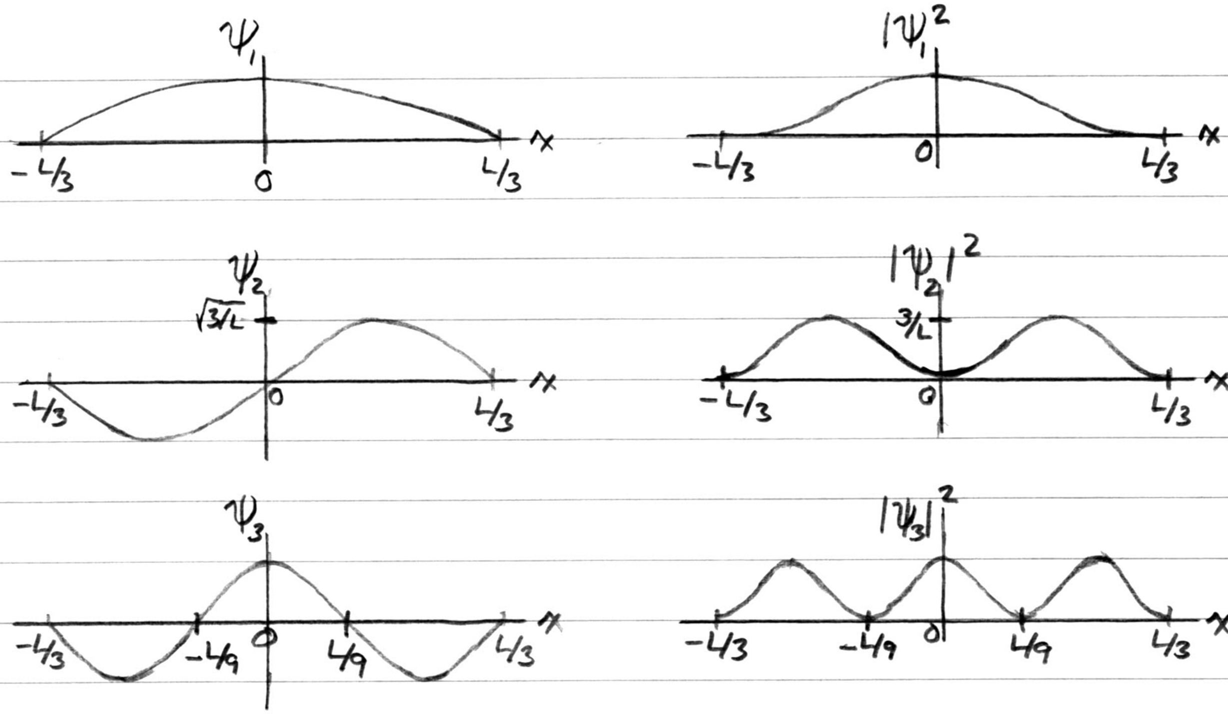
$$\text{and } |\psi_2(x)|^2 = \frac{3}{L}\sin^2\left(\frac{3\pi x}{L}\right)$$

$$\psi_3(x) = \sqrt{\frac{3}{L}} \sin\left(\frac{9\pi x}{2L} + \frac{3\pi}{2}\right) = (-)\sqrt{\frac{3}{L}} \cos\left(\frac{9\pi x}{2L}\right)$$

$$\text{and } |\psi_3(x)|^2 = \frac{3}{L}\cos^2\left(\frac{9\pi x}{2L}\right)$$

(Note the negative signs in front of ψ_2 and ψ_3 are of no physical consequence and can be ignored.)

b)



B. For an infinite square well, energy levels are $E_n = \frac{\hbar^2\pi^2 n^2}{2mL^2}$ $n = 1,2,3\ldots$

For a transition $n=2$ to $n=1$, the emitted photon has energy

$hf = E_2 - E_1$ but $f = c/\lambda$

$$\therefore \lambda = \frac{c}{f} = c\left(\frac{E_2-E_1}{h}\right)^{-1} = ch\left(\frac{\hbar^2\pi^2}{2mL^2}(2^2-1^2)\right)^{-1}$$

$$= \frac{2mL^2ch}{3\underbrace{\hbar^2\pi^2}_{(h/2)^2}} = \frac{8mL^2c}{3h}$$

$$\therefore L^2 = \frac{3h\lambda}{8mc}$$

$$\therefore L = \left(\frac{3h\lambda}{8mc}\right)^{1/2}$$

$$= \left(\frac{3\cdot 6.63\cdot 10^{-34}\,\mathrm{Js}\cdot 0.54\cdot 10^{-6}\,\mathrm{m}}{8\cdot 9.11\cdot 10^{-31}\,\mathrm{kg}\cdot 3\cdot 10^{8}\,\mathrm{m/s}}\right)^{1/2}$$

$$= 7.01\cdot 10^{-10}\ \mathrm{m}$$

C, (6.30) With $\psi_n(x) = \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L}$ for $0 < x < L$

and $\psi(x) = 0$ otherwise

$$\langle x \rangle_n = \int_{-\infty}^{\infty} dx\, \psi_n^*(x)\, x\, \psi_n(x) = \frac{2}{L}\int_0^L dx\, x \underbrace{\sin^2 \frac{n\pi x}{L}}_{\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)}$$

$$= \frac{1}{L}\int_0^L dx\, x - \frac{1}{L}\int_0^L dx\, x \cos\frac{2n\pi x}{L}$$ integrate by parts

$$= \frac{1}{L}\frac{x^2}{2}\Big|_0^L - \frac{1}{L}\Big[\underbrace{\frac{L}{2n\pi}x \sin\frac{2n\pi x}{L}\Big|_0^L}_{0} - \underbrace{\frac{L}{2n\pi}\int_0^L dx \sin\frac{2n\pi x}{L}}_{-\frac{L}{2n\pi}\cos\frac{2n\pi x}{L}\big|_0^L = 0}\Big]$$

$$= \frac{1}{L}\frac{L^2}{2} = \frac{L}{2}$$

$$\langle x^2 \rangle_n = \int_{\infty}^{\infty} dx\, \psi_n^*(x)\, x^2 \psi_n(x) = \frac{2}{L}\int_0^L dx\, x^2 \sin^2\frac{n\pi x}{L}$$

$$= \frac{2}{L}\int_0^L dx\, x^2 \cdot \frac{1}{2}\left(1 - \cos\left(\frac{2n\pi x}{L}\right)\right)$$

$$= \frac{1}{L}\int_0^L dx\, x^2 - \frac{1}{L}\int_0^L dx\, x^2 \cos\left(\frac{2n\pi x}{L}\right)$$ ← integrate second term by parts

$$= \frac{1}{L}\left(\frac{1}{3}L^3\right) - \frac{1}{L}\Big[\underbrace{\frac{L}{2n\pi}x^2 \sin\left(\frac{2n\pi x}{L}\right)\Big|_0^L}_{0} - \frac{L}{2n\pi}\int_0^L dx\,(2x)\sin\frac{2n\pi x}{L}\Big]$$

$$= \frac{1}{3}L^2 + \frac{1}{n\pi}\int_0^L dx\, x \sin\left(\frac{2n\pi x}{L}\right)$$ ← integrate by parts again...

$$= \frac{1}{3}L^2 + \frac{1}{n\pi}\Big[x\left(-\frac{L}{2n\pi}\right)\cos\frac{2n\pi x}{L}\Big|_0^L - \left(\frac{-L}{2n\pi}\right)\underbrace{\int_0^L dx \cos\frac{2n\pi x}{L}}_{\frac{L}{2n\pi}\sin\frac{2n\pi x}{L}\big|_0^L = 0}\Big]$$

$$= \frac{1}{3}L^2 + \frac{1}{n\pi}\left(\frac{-L}{2n\pi}\right)(L\underbrace{\cos 2n\pi}_{1} - 0)$$

$$= \frac{L^2}{3} - \frac{L^2}{2\pi^2 n^2}$$