Solution to Final Examination

MAT1320C, Fall 2014

Part I. Multiple-choice Questions (2 × 10 = 20 marks)

D, F, A, F, C, B, B, D, A, C

1. Evaluate $\int_0^{\pi/4} \sin^2 x \cos^3 x dx$.

(A) 0; (B) $-\frac{7\sqrt{2}}{120}$; (C) $-\frac{\sqrt{2}}{15}$; (D) $\frac{7\sqrt{2}}{120}$; (E) $\frac{5\sqrt{2}}{24}$; (F) $-\frac{5\sqrt{2}}{24}$.

Solution. (D) Let $u = \sin x$. Then $u' = \cos x$, $\cos^2 x = 1 - u^2$.

$$\int_0^{\pi/4} \sin^2 x \cos^3 x dx = \int_0^{\sqrt{2}/2} \sin^2 x \cos^3 x \left(\frac{1}{\cos x}\right) du = \int_0^{\sqrt{2}/2} u^2(1-u^2) du = \left[\frac{u^3}{3} - \frac{u^5}{5}\right]_{u=0}^{\sqrt{2}/2} = \frac{7\sqrt{2}}{120}.$$

2. Consider the function $f(x) = x^4 + 4x^3 + 6x^2 + 24x + 24$. Which one of the following statements is true?

(A) $f(x)$ attains a local maximum at $x = -1$.
(B) $f(x)$ attains a local minimum at $x = -1$.
(C) $x = -1$ is a critical number but $f(x)$ does not attain a local maximum or a local minimum at $x = -1$.
(D) $x = -1$ is not a critical number, but $f(x)$ has an inflection point at $x = -1$.
(E) $x = -1$ is a critical number and $f(x)$ has an inflection point at $x = -1$.
(F) $x = -1$ is not a critical number and $f(x)$ does not have an inflection point at $x = -1$.

Solution. (F) $f'(x) = 4x^3 + 12x^2 + 12x + 24$. $f'(-1) = -4 + 12 - 24 + 24 \neq 0$. $x = -1$ is not a critical number. (A), (B), (C) and (E) are false. $f''(x) = 12x^2 + 24x + 12 = 12(x^2 + 2x + 1) = 12(x + 1)^2$. $f''(-1) = 0$. However, when $x < -1$ or $x > -1$, $f''(x)$ is always positive, $f(x)$ does not have an inflection point at $x = -1$.

3. Suppose $h(x) = f(g(x))$. $g(2) = 3$, $g'(2) = 4$, $f(2) = 3$, $f'(2) = 6$, $f'(3) = 5$. Find $h'(2)$.

(A) 20; (B) 24; (C) 12; (D) 30; (E) 18; (F) 15.

Solution. (A) Since $g(2) = 3$, $h'(2) = f'(3)g'(2) = 20$.

4. What is $\int_0^1 \frac{x^4 + x^2 + 1}{x^2 + 1} dx$?

(A) $\ln 2$; (B) $\pi / 4$; (C) $1/3$;

(D) $(3\pi - 4) / 12$; (E) $(\pi + 4) / 6$; (F) $(3\pi + 4) / 12$.

Solution. (F) Use long division, $\dfrac{x^4 + x^2 + 1}{x^2 + 1} = x^2 + \dfrac{1}{x^2 + 1}$.

$$\int_0^1 \frac{x^4 + x^2 + 1}{x^2 + 1} dx = \int_0^1 \left(x^2 + \frac{1}{x^2 + 1} \right) dx = \left[\frac{x^3}{3} + \arctan x \right]_{x=0}^{1} = \frac{1}{3} + \frac{\pi}{4} = \frac{3\pi + 4}{12}.$$

5. On which interval(s) is the graph of the function $y = (x^2 - 7x + 14)e^x$ concave up?

(A) it never is; (B) $x > 2$; (C) $x < 1$ or $x > 2$;
(D) it always is; (E) $1 < x < 2$; (F) $x < 1$.

Solution. (C) $y' = (2x - 7)e^x + (x^2 - 7x + 14)e^x = (x^2 - 5x + 7)e^x$, and $y'' = (2x - 5)e^x + (x^2 - 5x + 7)e^x = (x^2 - 3x + 2)e^x$. Let $y'' = 0$. We have $x = 1$, $x = 2$. Since $y'' > 0$ when $x < 1$ or $x > 2$, y is concave up when $x < 1$ or $x > 2$.

6. If $f(x)$ is a continuous function such that $\int_1^7 f(x)dx = 9$, $\int_4^7 f(x)dx = 15$, what is $\int_1^4 (2f(x) - 5)dx$?

(A) -33; (B) -27; (C) -22; (D) 0; (E) 7; (F) 21.

Solution. (B) Since $\int_1^4 f(x)dx = \int_1^7 f(x)dx - \int_4^7 f(x)dx = -6$,

$$\int_1^4 (2f(x) - 5)dx = 2\int_1^4 f(x)dx - 5\int_1^4 dx = -12 - 15 = -27.$$

7. What is the equation of the tangent line to the curve $x^2y + 2xy - y^2 - 3x = -1$ at the point (1, 2)?

(A) $y = 3x - 1$; (B) $y = 5x - 3$; (C) $y = -\dfrac{1}{2}x + \dfrac{5}{2}$;

(D) $y = -2x + 4$; (E) $y = x + 1$; (F) $y = \dfrac{3}{2}x + \dfrac{1}{2}$.

Solution. (B) Take the derivative on both sides with respect to x. $2xy + x^2y' + 2y + 2xy' - 2yy' - 3 = 0$. At point (1, 2), $4 + y' + 4 + 2y' - 4y' - 3 = 0$. Then $y' = 5$. The equation of the tangent line has the form $y = 5x + b$. At point (1, 2), $2 = 5 + b$, $b = -3$. The equation is $y = 5x - 3$.

8. Which one of the following statements is always true?

(A) If a function $f(x)$ is continuous at $x = p$, then it is differentiable at p.

(B) If a function is continuous on an interval, then it has an absolute maximum on that interval.
(C) If function $f(x)$ has a local maximum at $x = p$, then $f'(p) = 0$.
(D) If a function is differentiable at $x = p$, then it is continuous at p.
(E) If $f''(p) = 0$, then $f(x)$ has an inflection point at $x = p$.
(F) If $x = p$ is a critical number of a function $f(x)$, then $f(x)$ attains a local maximum or a local minimum at p.

Answer. (D)

9. If $f(x) = (\sqrt{x})^{\sqrt{x}}$, the $f'(4) =$

(A) $1 + \ln 2$; (B) $4 \ln 2$; (C) 2; (D) 0; (E) $3 / 2$; (F) 4.

Solution. (A) Use logarithmic differentiation, $\ln f(x) = \sqrt{x}\ln\sqrt{x} = \frac{1}{2}\sqrt{x}\ln x$.

Then $(\ln f(x))' = \dfrac{f'(x)}{f(x)} = \dfrac{\ln x}{4\sqrt{x}} + \dfrac{1}{2\sqrt{x}} = \dfrac{1}{4\sqrt{x}}(\ln x + 2)$.

$f'(x) = (\sqrt{x})^{\sqrt{x}}\dfrac{1}{4\sqrt{x}}(\ln x + 2) = \dfrac{1}{4}\sqrt{x}^{\sqrt{x}-1}(\ln x + 2)$. $f'(4) = \dfrac{1}{4}2^{2-1}(2\ln 2 + 2) = \ln 2 + 1$.

10. Using the velocity data in the table below, estimate the distance the car travels in the 10 seconds with the Trapezoidal Rule.

t (sec)	0	2	4	6	8	10
v (m / sec)	32	24	20	17	11	7

(A) 158 m; (B) 176 m; (C) 183 m; (D) 196 m; (E) 203 m; (F) 208 m.

Solution. (C) The distance $D \approx \dfrac{2}{2}(32 + 2 \times 24 + 2 \times 20 + 2 \times 17 + 2 \times 11 + 7) = 183$.

Part II. Long-Answer Questions

11. (2 marks) If $F(x) = \int_{x^2}^{0} t(\arcsin t)^3\,dt$, what is $F'(x)$?

Solution. $F'(x) = -2x(x^2(\arcsin(x^2))^3) = -2x^3(\arcsin(x^2))^3$.

12. (2 marks) Newton's method is used to find an approximation of a root of the equation $x^3 - 3x + 1 = 0$ with $x_1 = 1.5$, find x_3 to 4 decimal places?

Solution. The iteration formula is $x_{i+1} = x_i - \dfrac{x^3 - 3x + 1}{3x^2 - 3}$. $x_2 = 1.5333$, and $x_3 = 1.5321$.

13. (2 marks) How many critical numbers does the function $f(x) = \dfrac{x^{1/3}}{x-1}$ have?

Solution. $f'(x) = \dfrac{\frac{1}{3}x^{-2/3}(x-1) - x^{1/3}}{(x-1)^2} = \dfrac{(x-1)-3x}{3x^{2/3}(x-1)^2} = \dfrac{-2x-1}{3x^{2/3}(x-1)^2}$.

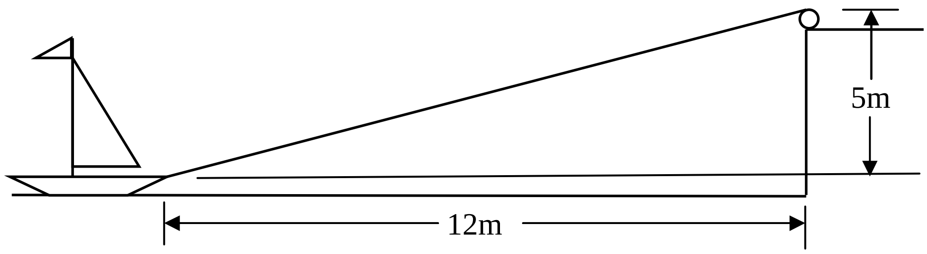
When $x = 0$ or $x = 1$, $f'(x)$ does not exist. When $x = -1/2$, $f'(x) = 0$. Since $x = 1$ is not in the domain of $f(x)$, this function has two critical numbers $x = -1/2$ and $x = 0$.

14. (2 marks) Find the linear approximation of the function $f(x) = \dfrac{1}{\sqrt{x+2}}$ near $x = 7$, and use it to approximate $\dfrac{1}{\sqrt{10}}$. Provide your answer to 4 decimal places.

Solution. $f'(x) = -\dfrac{1}{2}(x+2)^{-3/2}$. $f'(7) = -\dfrac{1}{54}$. Since $f(7) = \dfrac{1}{3}$, the linear approximation is $y = -\dfrac{1}{54}(x-7) + \dfrac{1}{3}$, or $y = -\dfrac{1}{54}x + \dfrac{25}{54}$.

When $x = 8$, $\dfrac{1}{\sqrt{x+2}} = \dfrac{1}{\sqrt{10}} \approx -\dfrac{8}{54} + \dfrac{25}{54} = \dfrac{17}{54} \approx 0.3148$.

15. (3 marks) The surface of a dock is 5 meters above the deck of a boat. The boat on the water is pulled in by a cable towards the dock. When the boat is 12 meters away horizontally from the dock, it is approaching the dock horizontally at a rate of 0.5 meters per second. How fast is the cable being pulled in?



Solution. Let the length of the cable be x and the horizontal distance between the boat and the dock be y. They are both functions of time t. Then $y^2 + 25 = x^2$. Taking the derivative on both sides of this relation to t, $2yy' = 2xx'$. $x' = yy'/x$. When $y = 12$, $x = 13$ and $y' = -0.5$, $x' = 12 \times (-0.5)/13 = -6/13$. The cable is pulled in at a rate 6 / 13 meter per second.

16. (3 marks) Use L'Hospital's Rule to evaluate the following limits:

(a) $\lim_{x\to 0}\frac{xe^x}{e^x-1}$.

Solution. $\lim_{x\to 0}\frac{xe^x}{e^x-1}=\lim_{x\to 0}\frac{e^x+xe^x}{e^x}=\lim_{x\to 0}(x+1)=1$.

(b) $\lim_{x\to 1}\left(\frac{x}{x-1}-\frac{1}{\ln x}\right)$.

Solution.

$$\lim_{x\to 1}\left(\frac{x}{x-1}-\frac{1}{\ln x}\right)=\lim_{x\to 1}\frac{x\ln x-(x-1)}{(x-1)\ln x}=\lim_{x\to 1}\frac{\ln x+1-1}{\ln x+\dfrac{x-1}{x}}=\lim_{x\to 1}\frac{x\ln x}{x\ln x+x-1}=\lim_{x\to 1}\frac{\ln x+1}{\ln x+1+1}=\frac{1}{2}.$$

17. (3 marks) The surface area and the volume of a circular cylinder with radius r and height h are given by $A = 2\pi r^2 + 2\pi rh$ and $V = \pi r^2 h$, respectively. If the surface area is fixed to be $2400\pi\,\text{cm}^2$, find the radius r and the height h that maximize the volume.

Solution. $2\pi r^2 + 2\pi rh = 2400\pi$. $r^2 + rh = 1200$. $rh = 1200 - r^2$. $V = \pi r(1200 - r^2)$, $0 < r < \infty$.

Let $V' = 1200\pi - 3\pi r^2 = 0$. $r^2 = 400$, $r = 20$. Then $h = 40$, and $V = 16000\pi$.

Since $V' > 0$ when $r < 20$, and $V' < 0$ when $r > 20$. Function V attains an absolute maximum at $x = 20$ and $h = 40$. The maximum volume of the circular cylinder is $16000\pi\,\text{cm}^3$.

18. (10 marks) Find the following:

(a) $\int_1^e \frac{1}{x\sqrt{1+\ln x}}\,dx$.

Solution. Let $u = 1 + \ln x$. Then $u' = \frac{1}{x}$.

$$\int_1^e \frac{1}{x\sqrt{1+\ln x}}\,dx=\int_1^2\frac{1}{\sqrt{1+\ln x}}\,du=\int_1^2 u^{-1/2}du=\left[2\sqrt{u}\right]_{u=1}^2=2(\sqrt{2}-1).$$

(b) $\int_1^e \sqrt{x}\ln x\,dx$.

Solution. By integration by parts, $\int \sqrt{x}\,dx=\frac{2}{3}x^{3/2}+C$.

$$\int_1^e \sqrt{x}\ln x dx = \int_1^e \ln x d\left(\frac{2}{3}x^{3/2}\right) = \left[\frac{2}{3}x^{3/2}\ln x\right]_{x=1}^e - \frac{2}{3}\int_1^e x^{3/2} d(\ln x) = \frac{2}{3}e^{3/2} - \frac{2}{3}\int_1^e x^{1/2}dx$$

$$= \frac{2}{3}e^{3/2} - \frac{2}{3}\left[\frac{2}{3}x^{3/2}\right]_{x=1}^e = \frac{2}{3}e^{3/2} - \frac{4}{9}(e^{3/2}-1) = \frac{2}{9}e^{3/2} + \frac{4}{9}.$$

(c) $\int \frac{5x+13}{x^2-2x-3}dx$.

Solution. Use partial fraction. $\frac{5x+13}{x^2-2x-3} = \frac{5x+13}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1} = \frac{A(x+1)+B(x-3)}{(x-3)(x+1)}$.

Then $A(x + 1)+ B(x - 3) = 5x + 13$.

Let $x = 3$. $4A = 28$, $A = 7$. Let $x = -1$. $-4B = 8$, $B = -2$.

$$\int \frac{5x+13}{x^2-2x-3}dx = 7\int \frac{1}{x-3}dx - 2\int \frac{1}{x+1}dx = 7\ln|x-3| - 2\ln|x+1| + C.$$

(d) $\int \frac{1}{(1-x^2)^{3/2}}dx$.

Solution. Let $x = \sin u$, $-\frac{\pi}{2} \le u \le \frac{\pi}{2}$. Then $(1 - x^2)^{3/2} = \cos^3 u$.

$$\int \frac{1}{(1-x^2)^{3/2}}dx = \int \frac{1}{\cos^3 u}(\cos u)du = \int \frac{1}{\cos^2 u}du = \tan u + C = \frac{x}{\sqrt{1-x^2}} + C.$$

(e) $\int \frac{2x+6}{x^2+4x+5}dx$.

Solution. Complete the square. $x^2 + 4x + 5 = (x + 2)^2 + 1$. Let $u = x + 2$. Then $x = u - 2$, and $2x + 6 = 2(u - 2) + 6 = 2u + 2$.

$$\int \frac{2x+6}{x^2+4x+5}dx = \int \frac{2u+2}{u^2+1}du = \int \frac{2u}{u^2+1}du + 2\int \frac{1}{u^2+1}du = \ln(u^2+1) + 2\arctan u + C$$

$= \ln(x^2 + 4x + 5) + 2\arctan(x + 2) + C$.

19. (3 marks) Sketch the graph of a function $y = f(x)$ that satisfies the following conditions:

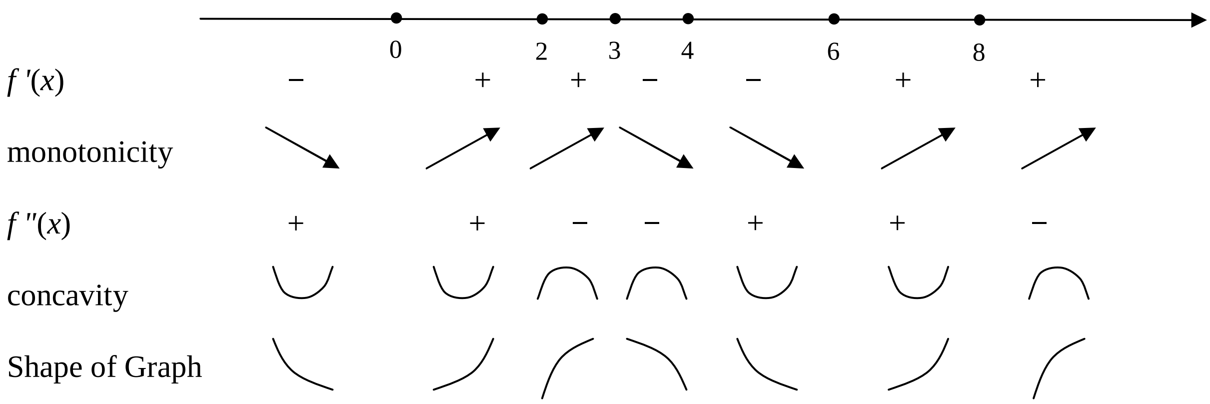
(a) $f(-1) = f(1) = f(5) = f(7) = f(9) = 0$; $f(0) = -3$;
(b) $f'(x) > 0$ for $0 < x < 3$, $6 < x < 8$, or $x > 8$; $f'(x) < 0$ for $x < 0$, or $3 < x < 6$;
(c) $f''(x) > 0$ for $x < 2$, or $4 < x < 8$; $f''(x) < 0$ for $2 < x < 4$, or $x > 8$;

(d) $\lim_{x \to 8^-} f(x) = \infty$; $\lim_{x \to 8^+} f(x) = -\infty$.

Label the special features (such as extrema, inflection points, asymptotes, etc.) on your graph.

Solution.

According to the information given by the problem, we have the following diagram:



Joining them together, we have the graph of $f(x)$: