— Calculus for the Life Sciences I —

MAT1330 DGD Workbook - Answer Key to Select Questions

- If you detect or suspect errors in the answers to this document, please let your professor know!

LEC 1 – High School Review.

1.3 # 20 Find the domain of $f(x) = \sqrt{2x-7}$. Ans: $[\frac{7}{2}, \infty)$

like 1.4 # 17 Find the formulas for the composition $f \circ g$ and $g \circ f$ and the product of the functions f and g; simplify where possible.

$$f(x) = \frac{x-1}{x+1} \qquad g(x) = 1/(2x)$$

Ans: $f \circ g(x) = \frac{1-2x}{1+2x}$ $\quad g \circ f(x) = \frac{x+1}{2x-2}$ $\quad fg(x) = \frac{x-1}{2x^2+2x}$

1.4 #33 Sketch the graph, state the domain and range, decide if the function has an inverse and if so, find it: $f(x) = \sqrt[3]{x} + 4$.

Ans: $D: (-\infty, \infty)$ $\quad R: (-\infty, \infty)$ $\quad f^{-1}(x) = (x-4)^3$

2.2 # 24 Solve $4e^{2x+1} = 20$. Ans: $x = \frac{1}{2}(\ln(5) - 1)$

2.2 #26 Solve $4e^{2x+3} = 7e^{3x-2}$. Ans: $x = \ln 4 + 5 - \ln 7$

2.2 #32 Express $y = 0.27^x$ in base e. Ans: $y = e^{(\ln(0.27)x}$

2.2 #42 Solve $\ln(\ln(x)) = 0$. Ans: $x = e$

LEC 2 – High School Review.

2.3 # 68 Graph the function. Give the average, max, min, amplitude, period and phase and mark them on the graph:

$$f(x) = 3 + 4\cos\left(2\pi\left(\frac{x-1}{5}\right)\right).$$

Ans: period $P = 5$ $\quad$ phase $\phi = 1$ $\quad$ amplitude $A = 4$ $\quad$ mean $M = 3$

min $M - A = -1$ $\quad$ max $M + A = 7$

Inequalities

Find all solutions to the inequality: $\frac{x+2}{2x-1} < \frac{1}{x+7}$. Ans: all $x \in (-7, \frac{1}{2})$

LEC 3 – Intro to DTDS.

3.1 #2 Write the updating function f associated with the following DTDS, and evaluate it at the given arguments. Is it a linear DTDS?

$$m_{t+1} = \frac{m_t^2}{m_t + 2}; \quad \text{Evaluate } f \text{ at } m_t = 0, m_t = 8, m_t = 20.$$

Ans: up. fun. $f(x) = \frac{x^2}{x+2}$ $\quad 0 \quad 6.4 \quad \frac{200}{11}$

3.1 #12 Write the updating function f associated with the following DTDS. Is it a linear DTDS?

$$M_{t+1} = 0.75M_t + 2$$

Determine the backward DTDS associated to the above DTDS. Use the backward DTDS to find the value M_0 (the value at the previous time step) given that $M_1 = 16$.

Ans: up. fun. $f(x) = 0.75x + 2$ (linear) $f^{-1}(x) = \frac{4}{3}x - \frac{8}{3}$ (up. fun. for backward DTDS) $M_0 = \frac{56}{3}$

3.1 #18 & 22 Graph some values of the following DTDS, starting with the given initial condition:

$$\ell_{t+1} = \ell_t - 1.7 \qquad \text{with initial value } \ell_0 = 13.1 \text{ cm}$$

Write down a formula for the general solution and sketch its graph. Sketch the graph of the updating function. Label the axes for each graph!

Ans: general solution $l_t = 13.1 - 1.7t$ graphical answers omitted

3.1 #19 & 23 Graph some values of the following DTDS, starting with the given initial condition:

$$n_{t+1} = 0.5n_t \qquad \text{with initial value } n_0 = 1200$$

Write down a formula for the general solution and sketch its graph. Sketch also the graph of the updating function.

Ans: up. fun. $f(x) = 0.5x$ general solution $x_t = (0.5^t)(1200)$ graphical answers omitted

LEC 4 – Fixed Points and Cobwebbing.

3.2 like #6 Given the DTDS governing the daily dose of a drug, $M_{t+1} = 0.75M_t + 2$, do three iterations of a cobweb starting at $M_0 = 16$ mg/L. Then plot the solution you found this way on a graph of t vs M_t. Compare with the general solution formula we proved in class.

Ans: omitted.

3.2 #8 & 26 Graph the updating function underlying the DTDS $z_{t+1} = 0.9z_t + 1$. Then cobweb four steps, starting from $z_0 = 3$. Label the axes!

Next: solve for all fixed points, and classify their stability using cobweb diagrams.

Ans: fixed point: $x^* = 10$ (stable) graphical answers omitted.

Ex. Graph the updating function underlying the DTDS $z_{t+1} = 1.1z_t - 1$. Then cobweb four steps, starting from $z_0 = 3$. Label the axes!

Next: solve for all fixed points, and classify their stability using cobweb diagrams.

Ans: fixed point: $x^* = 10$ (unstable) graphical answers omitted.

3.2 #12 & 30 Using the graph of the updating function underlying the DTDS $x_{t+1} = \dfrac{x_t}{x_t - 1}$, cobweb four steps, starting from $x_0 = 3$. (Restrict your updating function to the domain $x > 1$.) Label the axes!

Next: solve for all fixed points, and classify their stability using cobweb diagrams.

Ans: one fixed point: $x^* = 2$ (keep in mind: domain is restricted to $x > 1$)

Based on cobweb, solutions that start near $x^* = 2$ do not approach $x^* = 2$, so we consider this an unstable fixed point.

Note: textbook calls this "stable" because nearby solutions are not moving away from $x^* = 2$, but in MAT1330, we defined "stable" as "all nearby solutions must move closer to the fixed point".

Ex. Consider the DTDS $x_{t+1} = -\frac{1}{5}x_t^2 + 2x_t$. Cobweb this DTDS starting at $x_0 = 2$.

Next: solve for all fixed points, and classify them according to their stability.

Ans: two fixed points: $x^* = 0$ (unstable) $x^* = 5$ (stable)

LEC 5 – Stability of Fixed Points.

3.2 like #14 (or 3.4 #16) Given the following graph of f, which is the updating function of a DTDS, determine the number of fixed points of the DTDS and determine their stability using cobwebbing. Write you conclusions in full sentences.

Ans: two fixed points $x_1^* \approx 1.3$ (stable) $x_2^* \approx 5.5$ (unstable)

3.2 like #30 Find the equilibria of the following DTDS. Use cobwebbing to check each equilibrium for stability.

$$x_{t+1} = \frac{2x_t}{x_t - 1} \qquad (x > 1).$$

Ans: fixed point $x^* = 3$ (stable)

3.3 #28 In 1990 there were about 5000 southern mountain caribou in BC. In 2009, only about 1900 remained. Assume the annual per capita decline is constant. How long until the population falls below $m = 500$ (which is a level, below which it is expected the species will go extinct)?

Ans: when $t > \frac{19\ln(1/10)}{\ln(19/50)}$ years have passed since 1990 (1990 is the year for which $t = 0$)

3.4 #14 Find all nonnegative equilibria of the following DTDS, where a is some real positive parameter:

$$x_{t+1} = \frac{x_t}{a + x_t}.$$

(Extra bizarre: what happens if $a = 0$?)

Ans: fixed point(s) $x^* = 0$ and $x^* = 1 - a$. Since $a > 0$, we would need $a \leq 1$ in order for $x^* = 1 - a$ to be non-negative

If $a = 0$, then $f(x)$ would degenerate to $f(x) = 1$ for all x.

LEC 6 – Limits and Continuity.

4.2 (typical question) Sketch the graph and decide if the left-hand and right-hand limits at $a = 0$ exist, and then decide if the limit at $a = 0$ exists.

$$f(x) = \begin{cases} x + 1 & \text{if } x \leq 1 \\ x^2 & \text{if } x > 1 \end{cases}$$

Ans: $\lim x \to 0^- f(x) = 1$ $\lim x \to 0^+ f(x) = 1$ $\lim x \to 0 f(x) = 1$

Ans: $\lim x \to 1^- f(x) = 2$ $\lim x \to 1^+ f(x) = 1$ $\lim x \to 0 f(x)$DNE

4.2 like #13 Sketch the graph and decide if the left-hand and right-hand limits at $a = 0$ exist, and then decide if the limit exists.

$$f(x) = \begin{cases} |x + 1| + 1 & \text{if } x \leq 0 \\ |x - 2| & \text{if } x > 0 \end{cases}$$

Ans: $\lim x \to 0^- f(x) = 2$ $\lim x \to 0^+ f(x) = 2$ $\lim x \to 0 f(x) = 2$

Evaluate each of the following limits, showing all your steps:

4.2 #48 $\displaystyle\lim_{x\to 0}\frac{x^2-3x}{x^3-9x}$ Ans: $\frac{1}{3}$

4.2 #48 $\displaystyle\lim_{x\to 4}\frac{\sqrt{x}-2}{4-x}$ Ans: $-\frac{1}{4}$

4.4 #36 Sketch the graph and discuss continuity of

$$f(x)=\begin{cases}\dfrac{x^2-4}{x-2} & \text{if } x\neq 2\\[2ex] 0 & \text{if } x=2\end{cases}$$

Ans: f is discontinuous at $x=2$ because $\displaystyle\lim_{x\to 2}f(x)=4$ but $f(2)=0$.

LEC 7 – Infinite Limits & Limits at Infinity. Evaluate each of the following limits, showing all of your steps:

4.3 #12 $\displaystyle\lim_{x\to 1^+}\frac{x}{x^2-1}$ Ans: $+\infty$ (DNE)

4.3 #14 $\displaystyle\lim_{x\to -7^+}\sqrt{\frac{1}{x+7}}$ Ans: $+\infty$ (DNE)

4.3 #36 $\displaystyle\lim_{x\to\infty}0.7^x$ Ans: 0

4.3 #43 $\displaystyle\lim_{x\to\infty}\frac{x^3-6x+4}{3-x^3}$ Ans: -1

4.3 like #45 $\displaystyle\lim_{x\to\infty}\frac{(x-1)(x-3)(x-5)}{x^2-4}$ Ans: ∞ (DNE)

4.3 #50 $\displaystyle\lim_{x\to-\infty}\ln(3-x^3)$ Ans: ∞ (DNE)

4.4 # 6 Find a formula for a function $g(x)$ that makes the composition $\sin(g(x))$ discontinuous at $x=\pi$.

Ans: many answers possible. One possibility: $g(x)=\begin{cases}0 & \text{if } x>\pi\\ \pi/2 & \text{if } x\leq\pi\end{cases}$

Ex. (like Course Guide Lecture 7 question 7) Let $f(x)=\begin{cases}\dfrac{x^2-4x+3}{(x-1)^3} & \text{if } x\neq\pm 1\\[2ex] x+b & \text{if } x=\pm 1\end{cases}$

Find the limit of f as $x\to 1$ and as $x\to -1$. Is there a value of b that makes f continuous at $x=1$? Is there a value of b that makes f continuous at $x=-1$?

Ans: $\displaystyle\lim_{x\to 1^-}f(x)=-\infty$ so $\displaystyle\lim_{x\to 1}f(x)$ DNE. $f(1)=1+b$. There is no b that makes f continuous at $x=1$.

$\displaystyle\lim_{x\to -1^-}f(x)=-1$ $\displaystyle\lim_{x\to -1^+}f(x)=-1$ $\displaystyle\lim_{x\to -1}f(x)=-1$ $f(-1)=-1+b$

If $b=0$, then f would be continuous at $x=-1$.

LEC 8 – The Derivative: Definition & Basic Rules.

Let f be a function defined on a interval around x.

The **derivative** of f at x is, by definition,

Ans: $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ if this limit exists.

The **derivative of** $f(x)$ **at a point** a is, by definition,

Ans: $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ if this limit exists.

Using **the definition**, compute the derivative of each of the following functions:

a. $f(x) = 2 + \sqrt{3x+1}$ Ans: $f'(x) = \frac{3}{2\sqrt{3x+1}}$

b. $g(x) = \dfrac{3}{4+2x}$ Ans: $g'(x) = -\frac{6}{(4+2x)^2}$

c. $h(x) = \sqrt{x^2+1}$ Ans: $h'(x) = \frac{x}{\sqrt{x^2+1}}$

Using the rules of differentiation and simplifications where appropriate, compute the derivative of each of the following:

a. $\left(\dfrac{x^3}{x+1}\right)^{1/3}$ Ans: $\frac{2x+3}{3(x+1)^{4/3}}$

b. $\dfrac{x^2+\sqrt{x}}{x^3}$ Ans: $-x^{-2} - \frac{5}{2}x^{-7/2}$

c. $f(x) = \dfrac{1}{x^2+1}$ Ans: $\frac{-2x}{(x^2+1)^2}$

d. $g(x) = \dfrac{2}{x-3}$ Ans: $\frac{-2}{(x-3)^2}$

Ex. Discuss the differentiability of each of the following piece-wise functions and compute their derivatives if possible.

a. $f(x) = \begin{cases} 2x+2 & \text{if } x > 5 \\ 2x-3 & \text{if } x \leq 5. \end{cases}$

Ans: f is differentiable at all real numbers, except when $x = 5$

b. $g(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ x^3 & \text{if } x < 0. \end{cases}$

Ans: g is differentiable at all real numbers.

Ex. Suppose functions f and g are differentiable, and the following table of values is given.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
0	1	2	1	3
1	0	1	2	4
2	4	2	1	3
3	2	4	0	2
4	1	3	4	1

Determine the values of, and the derivatives of $f(x)g(x)$ and of $f(g(x))$, at the points $x = 0, 1, 2, 3, 4$.

Ans: $(fg)'(0) = 5$, $(fg)'(1) = 2$, $(fg)'(2) = 14$, $(fg)'(3) = 4$, $(fg)'(4) = 13$

$[f \circ g]'(0) = 3$, $[f \circ g]'(1) = 8$, $[f \circ g]'(2) = 3$, $[f \circ g]'(3) = 4$, $[f \circ g]'(4) = 3$

LEC 9 – Exponential & Log Derivatives.

5.1 #30 Let $g(t) = \sqrt{t}(t^2 - t - 1)$. Find $g'(t)$. Ans: $g'(t) = \frac{5}{2}t^{3/2} - \frac{3}{2}t^{1/2} - \frac{1}{2}t^{-1/2}$

5.1 #36 Let $f(t) = 2^{-4} + t^{-4} + e$. Find $f'(t)$. Ans: $f'(t) = -4t^{-5}$

5.1 #43 Find the equation of the tangent line to $y = f(x) = \sqrt[3]{x}$ when $x = 8$. Ans: $y - 2 = \frac{1}{12}(x - 8)$

Differentiate:

5.2 #10 $g(x) = ax^{3/4}e^{x-2}$ Ans: $\frac{3}{4}ax^{-1/4}e^{x-2} + ax^{3/4}e^{x-2}$

5.2 #18 $f(x) = \dfrac{2\sqrt{x}(x-1)e^{-x}}{9}$ Ans: $\frac{2}{9}\left[(\frac{3}{2}x^{1/2} - \frac{1}{2}x^{-1/2})e^{-x} + (x^{3/2} + x^{1/2})(-e^{-x})\right]$

Differentiate:

5.2 #24 $G(x) = \dfrac{(1+x)(2+x)}{(3+x)}$ Ans: $G'(x) = \frac{x^2+6x+7}{(x+3)^2}$

5.2 #27 $g(t) = \dfrac{e^t - 2t}{e^t + t}$ Ans: $g'(t) = \frac{3(t-1)e^t}{(e^t+t)^2}$

5.2 #40 Find $f'(0)$ if $f(x) = \dfrac{1+x^2}{e^x}$. Ans: $f'(0) = -1$

5.3 #4 Let $F(x) = \ln(x + \ln(x))$. Find $F'(x)$. Ans: $F'(x) = \frac{1+\frac{1}{x}}{1+\ln(x)}$

Differentiate each of the following:

5.3 #10 $f(x) = x^2\ln(x)$ Ans: $f'(x) = 2x\ln(x) + x$

5.3 #12 $F(y) = \dfrac{\ln(y)}{e^y}$ Ans: $F'(y) = \frac{1}{ye^y} - \frac{\ln(y)}{e^y}$

Differentiate each of the following:

5.3 #25 $f(x) = \sqrt{2 - \dfrac{x}{x-2}}$ Ans: $f'(x) = \frac{1}{(x-2)^2\sqrt{2-\frac{x}{x-2}}}$

5.3 #34 $f(x) = \ln x^4 + \ln^4 x$ Ans: $f'(x) = \frac{4}{x} + \frac{4}{x}(\ln(x))^3$

5.3 #41 Let $L(x) = \ln(\sqrt{\ln(x)})$. Find $L'(x)$. Ans: $L'(x) = \frac{1}{2x\ln(x)}$

5.3 #30 Let $h(x) = 2^x3^x$. Find $h'(x)$. Ans: $h'(x) = (\ln(6)) \cdot 6^x$

Differentiate:

$g(x) = 2^x3^{x^2}$ Ans: $g'(x) = (\ln 2 + 2x\ln 3) \cdot 2^x \cdot 3^{x^2}$

$f(x) = 2^x + 3^x$ Ans: $f'(x) = (\ln(2)) \cdot 2^x + (\ln(3)) \cdot 3^x$

$L(x) = 2^x3^{-x}$ Ans: $L'(x) = (\ln\left(\frac{2}{3}\right)) \cdot \left(\frac{2}{3}\right)^x$

Differentiate:

Ex. $F(x) = (2^x)^{\ln(x)}$ Ans: $f'(x) = (\ln(2)) \cdot \left(2^{x\ln(x)}\right) \cdot (\ln(x) + 1)$

bonus! $f(x) = x^x$. Ans: $f'(x) = (x^x)(\ln(x) + 1)$

LEC 10 – Trig, Inverse Trig & Implicit Differentiation.

5.3 #45 Let $F(x) = f(g(x))$ and let $H(x) = f(x)g(x)$.

If $g'(6) = -2$, $g(6) = 4$, $f(4) = \frac{1}{2}$, $f(6) = 5$, $f'(4) = 3$, and $f'(6) = 7$, find $F'(6)$ and $H'(6)$.
Ans: $F'(6) = -6 \quad H'(6) = 18$

5.3 #49 Differentiate $f(x) = (1+x)^{2+x}$ Ans: $f'(x) = (1+x)^{2+x}\left[\ln(x+1) + \frac{2+x}{1+x}\right]$

5.3: #34 Compute the derivative of $\sec(x)$ by first writing it in terms of $\sin(x)$ or $\cos(x)$.
Ans: use $\sec(x) = 1/\cos(x)$ to get $\frac{d}{dx}\left[\frac{1}{\cos(x)}\right] = \frac{\sin(x)}{\cos(x)\cos(x)} = \sec(x)\tan(x)$

5.3 #2 Find $g'(x)$ if $g(x) = \cos(x) + \cos(1)$ Ans: $g'(x) = -\sin(x)$

5.3 #4 Differentiate $f(x) = \sin(x) + \sin(2x) + \sin^2(x)$ Ans: $f'(x) = \cos(x) + 2\cos(2x) + 2\sin(x)\cos(x)$

Ex. Differentiate $h(x) = \cos^2(x^2)$ Ans: $h'(x) = -4x\cos(x^2)\sin(x^2)$

5.3 #12 Differentiate $f(x) = x\sec(x) + x\tan(x)$ Ans: $f'(x) = \sec(x) + x\sec(x)\tan(x) + \tan(x) + x\sec^2(x)$

5.3 #17 Find $f'(x)$ if $f(x) = \cos(\ln(x)) + \ln(\cos(x))$ Ans: $f'(x) = \frac{-\sin(\ln(x))}{x} - \tan(x)$

5.3 #22 Differentiate $g(u) = \dfrac{\sin(u) + \cos(u)}{\sin(u) - \cos(u)}$ Ans: $g'(u) = -\frac{2}{(\sin(u)-\cos(u))^2}$

5.3 #26 Differentiate $f(x) = e^{\cos(x)}$ Ans: $f'(x) = -\sin(x)e^{\cos(x)}$

5.3 #29 Differentiate $f(x) = 2^{3\tan(x)}$ Ans: $f'(x) = 3\ln(2)\sec^2(x)2^{3\tan(x)}$

5.3 #38 Find the equation of the tangent line to the graph of $f(x) = 1 + 2\sin(e^x)$ at the point where $x = 0$. Ans: $y = 2\cos(1)x + 1 + 2\sin(1)$

5.3 like #50 Let $f(x) = \arctan(2x-1)$. Find $f'(x)$. Ans: $f'(x) = \frac{2}{1+(2x-1)^2}$

5.3 #52 $g(x) = x^2\arcsin^2(x)$ Ans: $g'(x) = 2x\arcsin^2(x) + \frac{2x^2\arcsin(x)}{\sqrt{1-x^2}}$

5.3 #54 Find $\frac{dy}{dx}$ if $y = (\arctan(x^2)+1)^3$ Ans: $\frac{dy}{dx} = \frac{6x(\arctan(x^2)+1)^2}{1+x^4}$

5.3 #56 Find $\frac{dy}{dx}$ if $y = \arcsin(\sqrt[3]{x}) + \sqrt[3]{\arcsin x}$ Ans: $\frac{dy}{dx} = \frac{1}{3x^{2/3}\sqrt{1-x^{2/3}}} + \frac{1}{3(\arcsin(x))^{2/3}\sqrt{1-x^2}}$

5.5 (implicit differentiation) Suppose f is a function such that

$$f(x)e^x = xe^{f(x)}.$$

If $f(0) = 0$, find $f'(0)$. Ans: $f'(0) = 1$

5.5 #2 Given that y satisfies the following equation, use implicit differentiation to calculate y'.

$$e^{4x}y^4 - \sqrt{y} = 3x.$$

Ans: $y' = \frac{3-54^4e^{4x}}{4y^3e^{4x}-\frac{1}{2}y^{-3/2}}$

#8 If $f(x) - 2^{f(x)/2} = 4x$ and $f(0) = 2$, find $f'(0)$. Ans: $f'(0) = \frac{4}{1-\ln 2}$

#14 Given the equation $\ln(f(x)) = \ln(x^{3x-4})$, find $f'(x)$. Ans: $f'(x) = x^{3x-4}\left[3\ln(x) + \frac{3x-4}{x}\right]$

bonus! #59 Differentiate $y = \arcsin(x) + \arccos(x)$. Ans: $y' = 0$

LEC 11 – Graphing using Calculus.

Suppose a function $y = f(x), -\infty < x < \infty$, is continuous, with continuous first and second derivatives. Assume it satisfies the following conditions:

(1) $f'(x) < 0$ when $x < 0$, and $f'(x) > 0$ when $x > 0$

(2) $f''(x) < 0$ when $x < -2$, and $f''(x) > 0$ when $x > -2$

(3) $\lim_{x\to\infty} f(x) = \infty$, $\lim_{x\to-\infty} f(x) = 2$.

(4) $f(0) = -3$, $f(-2) = -1$.

(a) Where is the graph of $f(x)$ decreasing? Ans: f is decreasing on $(-\infty, 0)$

(b) Where is the graph of $f(x)$ concave up? Ans: f is concave up on $(-2, \infty)$

(c) Where does $f(x)$ attain a local maximum or minimum?
Ans: f attains a local (and global) min at $x = 0$

(d) What are the asymptotes of f? Ans: f has a H.A. $y = 2$ as $x \to -\infty$

(e) Sketch the graph of the function $y = f(x)$. (graphical solution omitted)

Ex. Consider the function $f(x) = \dfrac{1}{x^2} + \dfrac{1}{2x^3}$. Follow these steps to graph the function.

(a) Find the domain of f. Ans: $\{x \in \mathbb{R} : x \neq 0\}$

(b) Find the x-intercept(s) of f. Ans: $(-\frac{1}{2}, 0)$

(c) Calculate the derivative of f. Ans: $f'(x) = -2x^{-3} - \frac{3}{2}x^{-4}$

(d) Find the critical point(s) of f. Ans: $x = -\frac{3}{4}$ In fact, f has a local max at $(-\frac{3}{4}, \frac{16}{27})$

(e) Calculate the second derivative of f. Ans: $f''(x) = 6x^{-4} + 6x^{-5}$

(f) Find the point(s) of inflection. Ans: IP $(-1, \frac{1}{2})$

(g) Find the limits $\lim_{x\to 0^\pm} f(x)$. Ans: $\lim_{x\to 0^-} f(x) = -\infty$ and $\lim_{x\to 0^+} f(x) = +\infty$ (vertical asymptote)

(h) Find the limits $\lim_{x\to\pm\infty} f(x)$. Ans: $\lim_{x\to\pm\infty} f(x) = 0$ (horizontal asymptotes)

(j) Sketch the graph of f for $x \in [-2, 2]$. (graphical solution omitted)

LEC 12 – Extreme values.

Course Guide Lec 12: Question 1:
Find all local and global maxima and minima of the function $f(x) = \frac{1}{4}x^4 - \frac{1}{3}x^3 - \frac{15}{8}x^2 + \frac{2}{3}$ on the interval $[-3, 3]$.

Ans: local max at $(0, f(0))$ global min at $(\frac{5}{2}, f(\frac{5}{2}))$ local min at $(-\frac{3}{2}, f(-\frac{3}{2}))$
global max at $(-3, f(-3))$

Course Guide Lec 12: Question 4:

Find the global maximum and minimum of the function $f(x) = \dfrac{x^2 + 2x}{e^x}$ on the interval $x \in [0, 4]$.

Ans: local max at $(0, f(0))$ global min at $(-\sqrt{2}, f(-\sqrt{2}))$ global max at $(\sqrt{2}, f(\sqrt{2}))$

Course Guide Lecture 13, Question 9:
The oxygen concentration in a lake over a single day is given by the equation

$$C(t) = 10t^3 - 120t^2 + 210t + 12000,$$

where time, $0 \leq t \leq 24$, is measured in hours. When is the oxygen concentration highest? When is it lowest? What are the maximum and minimum values?

Ans: abs. min. concentration of 11020 when $t = 7$ abs. max. concentration of 86160 when $t = 24$

6.1 # 65 The Shannon Index measures the diversity of a species in an ecosystem. In the case of two species, it is defined by $H = -a\ln(a) - b\ln(b)$, where a is the percentage of species A and b is the percentage of species B. If there are just the two species, what is the maximum value and when does it occur? What does this mean in terms of the diversity of species in the ecosystem? Ans: global max of $H(\frac{1}{2}) = \ln 2$ occurs when $a = b = \frac{1}{2}$

LEC 13 – Optimization.

Course Guide Lec 13: Question 1:
A company harvests fish at some rate $h \geq 0$. The yield is $Y(h) = h(500 - h)$ tons of fish, the selling price is $200 per ton. The cost for harvesting at rate h is $C(h) = 1000h(1 + 0.1h)$ in dollars.
(a) Find the expression of the profit P (= revenue - cost) as a function of harvesting rate.
Ans: $P(h) = 99\,000h - 300h^2$

(b) Find the harvesting rate that maximizes profit. Ans: $h = 165$

(c) Find the maximum profit. Ans: $P(165) = \$8\,167\,500$

Course Guide Lecture 13: Question 6:
Find the point on the parabola $y = x^2$ that is the closest to the point $(1, 2)$ in the cartesian plane. Ans: the point $\left(\frac{1+\sqrt{3}}{2}, \left(\frac{1+\sqrt{3}}{2}\right)^2\right)$ on the parabola is closest to $(1, 2)$

6.2 #22 Calculate the maximum long-term harvest for a population satisfying the DTDS

$$N_{t+1} = \frac{rN_t}{1 + kN_t} - hN_t.$$

Suppose that $r = 1.5$ and $k = 1$.
(a) Find the equilibria as a function of h. Ans: $x^* = \frac{0.5-h}{h+1}$

(b) What range of h is associated to a positive equilibrium? Ans: $0 < h < 0.5$

(c)Find the harvest level giving maximum long-term harvest $Y(h)$ Ans: $h = -1 + \frac{1}{2}\sqrt{6}$

(d) Sketch the graph of $Y(h)$ and compute the max value. (graphical solution omitted)

Ans: max harvest is $Y(-1 + \frac{1}{2}\sqrt{2}) \approx 0.05051$

LEC 14 – L'Hopital's Rule. Evaluate each of the following limits:

a. $\displaystyle\lim_{x\to 0} \frac{e^x - 1 - x}{\cos(x) - 1}$ Ans: -1

b. $\displaystyle\lim_{x\to 1} \frac{3(x-1)^2}{e^{2x-2} - x^2}$ Ans: 3

c. $\displaystyle\lim_{x\to 1^+} \frac{3x^2}{e^{2x-2} - x^2}$ Ans: $+\infty$

d. $\displaystyle\lim_{x\to 0} \frac{\arctan(x)}{x}$ Ans: 1

e. $\lim_{x\to\infty} \dfrac{e^x - 2}{3 - 2e^x}$ Ans: $-\frac{1}{2}$

f. $\lim_{x\to 0} \dfrac{\sin(x) - x}{x^3}$ Ans: $-\frac{1}{6}$

g. $\lim_{x\to 0} \dfrac{1 - e^x}{1 - e^{x/2}}$ Ans: 2

h. $\lim_{x\to(\pi/2)^+} \dfrac{\cos(x)}{(x - \pi/2)^2}$ Ans: $-\infty$

i. $\lim_{x\to\infty} \dfrac{\ln(x^6)}{x^6}$ Ans: 0

j. $\lim_{x\to\pi} \cot^2(x)(x - \pi)^2$ Ans: 1

k. $\lim_{x\to 0^-} -x^{-2}e^{1/x}$ Ans: 0

l. $\lim_{x\to-\infty} x^2 e^x$ Ans: 0

m. $\lim_{x\to\infty} x^3 e^{-x^2}$ Ans: 0

n. $\lim_{x\to\infty} x\tan(\frac{1}{x})$ Ans: 1

o. $\lim_{x\to\infty} \left(\sqrt{x^2 + x} - \sqrt{x^2 - 2x}\right)$ Ans: $\frac{3}{2}$

p. $\lim_{x\to 0^+} \left[\dfrac{1}{x} - \dfrac{\ln(1 + x)}{x^2}\right]$ Ans: $\frac{1}{2}$

q. $\lim_{x\to\infty} \left(\sqrt{x - 1} - \sqrt{x + 3}\right)$ Ans: 0

r. $\lim_{x\to\infty} \left(xe^{\frac{1}{x}} - x\right)$ Ans: 1

s. $\lim_{x\to\infty} (x + 3)^{1/x}$ Ans: 1

t. $\lim_{x\to\infty} (\ln x)^{1/x}$ Ans: 1

u. $\lim_{x\to 0} (1 + 2x)^{\frac{1}{x}}$ Ans: e^2

v. $\lim_{x\to\infty} \left(\dfrac{x}{x + 1}\right)^x$ Ans: e^{-1}

w. $\lim_{x\to 0^+} (\sin x)^{\tan x}$ Ans: 1

LEC 15 – Taylor Polynomials. Recall: the Taylor polynomial of $f(x)$ centred at the base point a is

$$T_n(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f^{(3)}(a)}{3!}(x - a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n$$

Course Guide Question 1: Consider the function $f(x) = (3x + 5)^{4/3}$.

- Find the Taylor polynomial T_3 using base point $a = 1$ for the function $f(x)$.

  Ans: $T_3(x) = 16 + 8(x - 1) + \frac{1}{2}(x - 1)^2 - \frac{1}{24}(x - 1)^3$

- Evaluate the error in the approximation by calculating $|f(0.8) - T_3(0.8)|$ to six decimal places. Ans: $|f(0.8) - T_3(0.8)| = 0.000010857...$

Course Guide Question 9: Consider the function $f(x) = 1 + \sin(2x - 2)$.
(a) Use a linear approximation of f to estimate the value of $f(0.9)$.

Ans: $L(x) = T_1(x) = 1 + 2(x - 1)$ $f(0.9) \approx L(0.9) = 0.8$

(b) Justify from the graph of f why the approximation of $f(0.9)$ in (a) is below the actual value. (graphical solution omitted)

(c) Use a Taylor polynomial of degree 3 to approximate $f(0.9)$. Ans: using centre $a = 1$, we have $T_3(x) = 1 + 2(x - 1) - \frac{4}{3}(x - 1)^3$ and $f(0.9) \approx T_3(0.9) = 0.8013333...$

LEC 16 – Stability of DTDS. Recall this important theorem!

The Stability Theorem (Derivative Test for Stability) Suppose x^* is a fixed point (equilibrium) of a DTDS $x_{t+1} = f(x_t)$. Then the fixed point x^* is

- stable if $|f'(x^*)| < 1$
- unstable if $|f'(x^*)| > 1$

Course Guide Question 3: The density of fish (i.e. number of fish per cubic metre) in a lake is determined by the discrete-time dynamical system

$$x_{t+1} = \frac{4x_t}{1 + 3x_t^2}$$

where t is the time in years since the beginning of the observation. Initially, the density is $x_0 = 0.5$.

(a) What will the density be after three years? (4 decimal places are enough) Ans: $x_4 \approx 0.9824$

(b) What is the updating function $f(x)$? Ans: $f(x) = \frac{4x}{1+3x^2}$

(c) What are the biologically relevant equilibria? Ans: $x^* = 0$, $x^* = 1$

(d) Use the derivative test to determine the stability properties for each of the two equilibria.

Ans: $x^* = 0$ is unstable, $x^* = 1$ is stable

Course Guide Question 4 Consider the DTDS $x_{t+1} = f(x_t)$, where the updating function is $f(x) = \dfrac{1 + x}{1 + x^2}$.

(a) Find the equilibrium point(s). Ans: $x^* = 1$

(b) Use the derivative test to evaluate the stability of each equilibrium point. Ans: $x^* = 1$ is stable

(c) Starting from $x_0 = 5$, calculate x_1, x_2, x_3. Ans: $x_1 \approx 0.2307$, $x_2 \approx 1.1685$, $x_3 \approx 0.9167$

(d) Sketch the graph of the updating function and use cobwebbing to confirm your calculations.

(graphical solution omitted)

Course Guide Question 4 Consider the DTDS $x_{t+1} = f(x_t)$, where the updating function is $f(x) = \dfrac{5x}{1 + 4x^2}$.

(a) Find the equilibrium point(s). Ans: $x^* = -1$, $x^* = 0$, $x^* = 1$

(b) Use the derivative test to evaluate the stability of each equilibrium point.

Ans: $x^* = -1$ is stable (but may not be biologically relevant), $x^* = 0$ is unstable, $x^* = 1$ is stable

(c) Starting from $x_0 = 5$, calculate x_1, x_2, x_3. Ans: $x_1 \approx 0.2475$, $x_2 \approx 0.9940$, $x_3 \approx 1.003$

(d) Sketch the graph of the updating function and use cobwebbing to confirm your calculations. (graphical solution omitted)

Course Guide Question 4 Consider the DTDS $x_{t+1} = f(x_t)$, where the updating function is $f(x) = rxe^{-x}$ where r denotes a positive parameter.

(a) Find the equilibrium point(s). Ans: $x^* = 0$ and $x^* = \ln(r)$

(b) Use the derivative test to evaluate the stability of each equilibrium point.

Ans: $x^* = 0$ is stable if $|r| < 1$ and unstable if $|r| > 1$
$x^* = \ln(r)$ is stable if $1 < r < e^2$ and unstable if $r < 1$ or $r > e^2$

(c) Starting from $x_0 = 5$, calculate x_1, x_2, x_3.
Ans: Using $r = 3$ and $f(x) = 3xe^{-x}$, $x_1 \approx 0.1011$, $x_2 \approx 0.2741$, $x_3 \approx 0.6251$

(d) Sketch the graph of the updating function and use cobwebbing to confirm your calculations.
(graphical solution omitted)

Course Guide Question 4 Consider the DTDS $x_{t+1} = f(x_t)$, where the updating function is $f(x) = \dfrac{2x}{1 + 0.1x}$.

(a) Find the equilibrium point(s). Ans: $x^* = 0$ and $x^* = 10$

(b) Use the derivative test to evaluate the stability of each equilibrium point.

Ans: $x^* = 0$ is unstable and $x^* = 10$ is stable

(c) Starting from $x_0 = 5$, calculate x_1, x_2, x_3. Ans: $x_1 \approx 6.667$, $x_2 \approx 8$, $x_3 \approx 8.889$

(d) Sketch the graph of the updating function and use cobwebbing to confirm your calculations.
(graphical solution omitted)

LEC 17 – Newton's Method. Recall: The **Intermediate Value Theorem** says that if f is continuous on a closed interval $[a, b]$ and y is between $f(a)$ and $f(b)$ then there is an $x \in [a, b]$ such that $f(x) = y$.

Recall: Newton's method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Course Guide Question #5: Complete the following steps to estimate the solution of the equation

$$\sin\left(x + \frac{\pi}{2}\right) = \frac{x}{2}$$

(a) Use the Intermediate Value Theorem to explain why we know that there is a solution between 0 and $\frac{\pi}{2}$. Ans: Using $f(x) = \sin(x + \frac{\pi}{2}) - \frac{x}{2}$, we see that $f(x)$ is continuous on $[0, \frac{\pi}{2}]$ and $f(\frac{\pi}{2}) < 0 < f(0)$. Since $Y = 0$ is between $f(\frac{\pi}{2})$ and $f(0)$, the IVT guarantees some number $c \in (0, \frac{\pi}{2})$ such that $f(c) = Y$.

(b) Perform three iterations of Newton's method with the initial value $x_0 = \frac{\pi}{4}$ (use 8 decimal places). Ans: $x_1 \approx 1.045862029$, $x_2 \approx 1.02991391668$, $x_3 \approx 1.029866529$

Course Guide: Question 2: Consider the DTDS $N_{t+1} = \ln(3 - N_t^2)$. (a) Use the Intermediate Value Theorem to show that there is an equilibrium in the closed interval $[0, 1]$. Ans: Using $g(x) = \ln(3 - x^2) - x$, the roots of $g(x)$ correspond to solutions to the equation $x = f(x)$. Since $Y = 0$ is between $g(0)$ and $g(1)$ and since $g(x)$ is continuous on $[0, 1]$, the IVT guarantees that there is some number $c \in (0, 1)$ such that $g(c) = Y$. Consequently, this number c is a solution to the equation $x = f(x)$, hence is a fixed point for this DTDS.

(b) Use Newton's method to solve for the equilibrium up to four iterations using the initial guess 1. Please give decimal 4 points for your calculation. Ans: $x_0 = 1$, $x_1 \approx 0.84657359$, $x_2 \approx 0.834546205$, $x_3 \approx 0.834486867$, $x_4 \approx 0.834486865$

Course Guide: Question 4: The goal of this question is to show that the function $f(x) = x^3 + x^2 + 3x + 2$ for $x \in (-\infty, \infty)$ has exactly one zero.
(a) Use the intermediate value theorem to show that there exists (at least) one zero. Ans: $f(-1) = -1$ and $f(0) = 2$ (for example). Since $Y = 0$ is between $f(-1)$ and $f(0)$, and since $f(x)$ is continuous on $[-1, 0]$, the IVT guarantees that there is some number $c \in (-1, 0)$ such that $f(c) = Y$.

(b) Use the Mean Value Theorem to show that if there are two (or more) zeros, then there is at least one critical point. Ans: Assume there exist two numbers a, b such that $a \neq b$ and $f(a) = f(b) = 0$. Since $f(x)$ is differentiable for all real numbers, $f(x)$ is differentiable on $[a, b]$. The MVT guarantees there exists a number $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a} = \frac{0-0}{b-a} = 0$. Thus, the number c guaranteed by MVT would be a critical number of $f(x)$.

(c) Show that the function f does not have a critical point. Ans: $f'(x) = 3x^2 + 2x + 3$ and $0 = f'(x)$ has no solutions, as can be seen by the discriminant of the quadratic equation $0 = 3x^2 + 2x + 3$. Thus, f has no "type 1" critical numbers. Furthermore, the domain of f' is the same as the domain of f so there are no "type 2" critical numbers. Thus, f has no critical numbers.

(d) Put all your arguments together to show that the function has exactly one zero. Ans: Since the assumption that f has more than one zero leads to the contraction that f has a critical number (which we know to be false), f cannot have more than one zero. However, f has at least one zero by (a), hence has exactly one zero.

(e) Find the root, accurate to 3 decimal places, using Newton's method. Ans: With initial guess $x_0 = -0.5$, we get $x_1 \approx -0.7272727$, $x_2 \approx -0.715279444$, $x_3 \approx -0.71522524$, $x_4 \approx -0.715225238$, so the root is $x \approx -0.715$

LEC 18 – Antiderivatives. Compute the following:

$$\int \sin(x)\, dx$$

Ans: $-\cos(x) + C$

$$\int \cos(x)\, dx$$

Ans: $\sin(x) + C$

$$\int \sec^2(x)\, dx$$

Ans: $\tan(x) + C$

$$\int \sec(x)\tan(x)\, dx$$

Ans: $\sec(x) + C$

$$\int \frac{1}{1+x^2}\, dx$$

Ans: $\arctan(x) + C$

$$\int \frac{1}{\sqrt{1-x^2}}\, dx$$

Ans: $\arcsin(x) + C$

$$\int x^3\, dx$$

Ans: $\frac{1}{4}x^4 + C$

$$\int \sqrt{x}\, dx$$

Ans: $\frac{2}{3}x^{3/2} + C$

$$\int 17x^2\, dx$$

Ans: $\frac{17}{3}x^3 + C$

$$\int \frac{31}{x^2}\, dx$$

Ans: $-31x^{-1} + C$

$$\int \frac{8}{x}\, dx$$

Ans: $8\ln|x| + C$

$$\int 7e^x\, dx$$

Ans: $7e^x + C$

Find the following indefinite integrals.

(a) $\int \frac{(1+\sqrt{x})^2}{x^2} dx$ Ans: $-\frac{1}{x} - \frac{4}{\sqrt{x}} + \ln|x| + C$

(b) $\int \frac{(t+1)^2}{2t^3}\, dt$ Ans: $\frac{1}{2}\ln(t) - \frac{1}{t} - \frac{1}{4t^2} + C$

(c) $\int \frac{(\sqrt{x}+2)^2}{x^2}\, dx$ Ans: $\ln|x| - \frac{8}{\sqrt{x}} - \frac{4}{x} + C$

(d) $\int \frac{(2-x)^2}{x}\, dx$ Ans: $4\ln|x| - 4x + \frac{1}{2}x^2 + C$

(e) $\int \left(10x^4 - \frac{2}{x} + \frac{4}{\sqrt[3]{x}} - 1\right)\, dx$ Ans: $2x^5 - 2\ln|x| + 6x^{2/3} - x + C$

Find the value of $F(1)$ when $F(0) = 1$ and $F'(t) = f(t)$ is given by $f(t) = 3t^3 + 1$. Ans: $F(t) = \frac{3}{4}t^4 + t + 1$ and $F(1) = \frac{11}{4}$

7.1 #2,4,6 Which of the following differential equations are pure-time? Which are autonomous?

$dy/dt = 2y$ Ans: autonomous

$y' = 2xe^x$ Ans: pure-time

$df/dx = \ln(x) + x - 1$ Ans: pure-time

$df/dt = 3t^3 f(t)$ Ans: neither

7.2 #42 Suppose organisms grow in mass according to the differential equation

$$\frac{dM}{dt} = \alpha t^n.$$

where M is measured in grams and t in days. Suppose $n = -1/2$ and $\alpha = 2$.

(a) Find the units of α. Ans: grams/$\sqrt{\text{day}}$

(b) Suppose $M(0) = 5$ g. Find the solution. Ans: $M(t) = 4\sqrt{t} + 5$

(c) Sketch the graphs of $M(t)$ and $M'(t)$. (graphical solution omitted)

(d) Describe your results in words. Ans: The rate of growth decreases as t increases, but it always positive, so the mass is always increasing, though at a slower and slower rate.

LEC 19 – Substitution.

1. $\int (17x - 17)^{17}\, dx.$ Ans: $\frac{1}{306}(17x - 17)^{18} + C$

2. $\int \sqrt[3]{12x + 2014}\, dx.$ Ans: $\frac{1}{16}(12x + 2014)^{4/3} + C$

3. $\int \frac{t}{(1+t^2)^2} dt$ Ans: $-\frac{1}{2}(1+t^2)^{-1} + C$

4. $\int \frac{(\ln(x))^3}{x}\, dx$ Ans: $\frac{1}{4}(\ln(x))^4 + C$

5. $\int \frac{3x+1}{(3x^2+2x+1)^6}\, dx$ Ans: $-\frac{1}{10}(3x^2 + 2x + 1)^{-5} + C$

6. $\int \sin(x)e^{\cos(x)}\, dx$ Ans: $-e^{\cos(x)} + C$

7. $\displaystyle\int \frac{\cos(x)}{\sqrt{\sin(x)}}\,dx$ Ans: $2\sqrt{\sin(x)}+C$

8. $\displaystyle\int 5^{2x-3}\,dx.$ Ans: $\frac{1}{2\ln(5)}5^{2x-3}+C$

9. Find $\displaystyle\int \frac{5}{1+9x^2}\,dx.$ [Hint: differentiate $g(x)=\arctan(x)$.] Ans: $\frac{5}{3}\arctan(3x)+C$

10. $\displaystyle\int \frac{(\ln(x))^3}{3x}\,dx$ Ans: $\frac{1}{12}(\ln(x))^4+C$

11. $\displaystyle\int \frac{3x+1}{(3x^2+2x+1)^6}\,dx$ Ans: $-\frac{1}{10(3x^2+2x+1)}+C$

12. $\displaystyle\int \frac{t}{(1+t^2)^2}\,dt$ Ans: $-\frac{1}{2(1+t^2)}+C$

13. $\displaystyle\int \frac{e^x}{e^x+1}\,dx$ Ans: $\ln|e^x+1|+C$

14. $\displaystyle\int \frac{(\ln(z))^2}{z}\,dz$ Ans: $-\frac{1}{3}(\ln(z))^3+C$

15. $\displaystyle\int \frac{\sin(\frac{1}{x})}{x^2}\,dx$ Ans: $\cos(\frac{1}{x})+C$

16. $\displaystyle\int e^{3x}\sqrt{2-e^{3x}}\,dx$ Ans: $-\frac{2}{9}(2-e^{3x})^{3/2}+C$

17. Find the value of $F(1)$ when $F(0)=1$ and $F'(t)=f(t)$ is given by

    (a) $f(t)=\dfrac{1}{17t+12}$ Ans: $F(t)=\frac{1}{17}\ln|17t+12|+\frac{1}{17}\ln(12)-1$

    (b) $f(t)=12e^{2t}$ Ans: $F(t)=12e^{t}-11$

18. Find the following indefinite integrals

    (a) $\displaystyle\int \frac{\tan(x)}{\ln(\cos(x))}\,dx$ Ans: $-\ln|\ln(\cos(x))|+C$

    (b) $\displaystyle\int \sin(x)e^{\cos(x)}\,dx$ Ans: $-e^{\cos(x)}+C$

    (c) $\displaystyle\int \frac{\cot(x)}{\ln(\sin(x))}\,dx$ Ans: $\ln|\ln(\sin(x))|+C$

    (d) $\displaystyle\int \frac{\cos(\ln(x))}{x}\,dx$ Ans: $\sin(\ln(x))+C$

    (e) $\displaystyle\int \frac{e^x+1}{e^x+x}\,dx$ Ans: $\ln|e^x+1|+C$

    (f) $\displaystyle\int \frac{\sin(x)}{1+\cos^2(x)}\,dx$ Ans: $-\arctan(\cos(x))+C$

    (g) $\displaystyle\int \frac{\cos(x)}{(\sin^2(x))^{1/3}}\,dx$ Ans: $3(\sin(x))^{1/3}+C$

    (h) $\displaystyle\int \cos(x)e^{\sin(x)}\,dx$ Ans: $e^{\sin(x)}+C$

(i) $\displaystyle\int \frac{e^x+2}{e^x+2x}\,dx$ Ans: $\ln|e^x+2x|+C$

(j) $\displaystyle\int \frac{(\ln(x))^3}{x}\,dx$ Ans: $\frac{1}{4}(\ln(x))^4+C$

(k) $\displaystyle\int \left(\frac{2}{x(1+\ln(x))}\right)\,dx$ Ans: $-2\ln|1-\ln(x)|+C$

(l) $\displaystyle\int \frac{\cos(x)}{\sqrt{\sin(x)}}\,dx$ Ans: $2\sqrt{\sin(x)}+C$

19. Find the anti-derivative $F(x)$ of $f(x)=\dfrac{e^{\arcsin(x)}}{\sqrt{1-x^2}}$ such that $F(0)=1$. Ans: $F(x)=e^{\arcsin(x)}+0$

LEC 20 – Integration By Parts.

1. $\displaystyle\int 16x^3\ln(7x)dx$ Ans: $4x^4\ln(7x)-x^4+C$

2. $\displaystyle\int \arcsin(x)\,dx.$ Ans: $x\arcsin(x)+\sqrt{1-x^2}+C$

3. $\displaystyle\int 3x^2\cos(0.5x)\,dx$ Ans: $6x^2\sin(0.5x)+24x\cos(0.5x)-48\sin(0.5x)+C$

4. Find the value of $F(1)$ when $F(0)=1$ and $F'(t)=f(t)$ is given by $f(t)=(t+t^2)e^{-t}$.
Ans: $F(t)=-e^{-t}(t+t^2)+(1+2t)e^{-t}-2e^{-t}+2$

5. Find the function $f(x)$, such that $f''(x)=\ln(x)$ and $f(1)=f'(1)=0$.
Ans: $f(x)=\frac{1}{2}x^2\ln(x)-\frac{1}{4}x^2+\frac{1}{4}$

6. Find the following indefinite integrals.

(a) $\displaystyle\int (x+1)\sin(x)\,dx$ Ans: $-(x+1)\cos(x)+\sin(x)+C$

(b) $\displaystyle\int x^2\cos(x)\,dx$ Ans: $x^2\sin(x)+2x\cos(x)-2\sin(x)+C$

(c) $\displaystyle\int 16x^3\ln(7x)dx$ Ans: $4x^2\ln(7x)-x^4+C$

7. Find the indefinite integral of each of the following functions. Check your results by differentiating.

(a) $f(x)=\frac{x}{2}\cos(5x)$ Ans: $-\frac{1}{10}\sin(5x)+\frac{1}{50}\cos(5x)+C$

(b) $f(x)=\sqrt{x}\ln x$ Ans: $\frac{2}{3}x^{3/2}\ln(x)-\frac{4}{9}x^{3/2}+C$

(c) $f(x)=\arcsin(x)$ Ans: $x\arcsin(x)-\sqrt{1-x^2}+C$

(d) $f(x)=x3^x$ Ans: $\frac{x3^x}{\ln(3)}-\frac{1}{\ln(3)}3^x+C$

(e) $f(x)=x^2e^{-x}$ Ans: $-x^2e^{-x}-2xe^{-x}+2e^{-x}+C$

8. Find the value of $F(1)$ when $F(0)=1$ and $F'(t)=f(t)$ is given by

(a) $f(t)=(t+t^2)e^{-t}$

Ans: $F(t)=-e^{-t}(t^2-t+1)+2$

(b) $f(t)=3t\cos(t^2)$

Ans: $F(t)=\frac{3}{2}\sin(t^2)+1$

9. Find the following indefinite integrals

   (a) $\int x \ln(x)\, dx$ Ans: $\frac{1}{2}x^2 \ln(x) - \frac{1}{4}x^2 + C$

   (b) $\int (x+1)\sin(x)\, dx$ Ans: $-(x+1)\cos(x) + \sin(x) + C$

   (c) $\int (x+1)\cos(x)\, dx$ Ans: $x^2 \sin(x) + 2x\cos(x) - 2\sin(x) + C$

   (d) $\int (x+1)\ln(x)\, dx$ Ans: $(\frac{1}{2}x^2 + x)\ln(x) - \frac{1}{4}x^2 - x + C$

   (e) $\int (x-2)\sin(x)\, dx$ Ans: $-(x-2)\cos(x) + \sin(x) + C$

   (f) $\int \sqrt[3]{x} \ln(x)\, dx$ Ans: $\frac{3}{4}x^{4/3}\ln(x) - \frac{9}{16}x^{4/3} + C$

   (g) $\int 3x^2 \cos(0.5x)\, dx$ Ans: $6x^2 \sin(0.5x) + 24x\cos(0.5x) - 48\sin(0.5x) + C$

10. Find the function $f(x)$, such that $f''(x) = \ln(x)$ and $f(1) = f'(1) = 0$.
    Ans: $f(x) = \frac{1}{2}x^2 \ln(x) - \frac{1}{4}x^2 + \frac{1}{4}$

11. Let $V(t)$ be the volume of a benign tumour in cm^3 after t years. For $t \geqslant 0$, suppose that $V(t)$ satisfies the following differential equation

$$\frac{dV}{dt} = (1+t)e^{-t}.$$

   a. If initially $V(0) = 1$, find $V(t)$. Ans: $V(t) = -(1+t)e^{-t} - e^{-t} + 3$

   b. Compute $\lim_{t\to\infty} V(t)$ and interpret.
      Ans: 3 In the long run, the tumour's volume approaches 3 cm^3.

   c. Use Newton's method to find when the volume of the tumour will be 2 cm^3. Use 5 decimal places in your computations and find the answer with 3 decimal places of precision.
      Ans: Solutions to $V(t) = 2$ correspond to solutions to $0 = g(x)$ for $g(x) = -e^{-x}(2+x) + 1$. Using Newton's method to find roots of $g(x)$ and using the initial guess $x_0 = 1$, we get $x_1 \approx 1.1408590$, $x_2 \approx 1.146185641$, $x_3 \approx 1.146193221$, $x^4 \approx 1.14693221$, so $v(t) = 2$ when $t \approx 1.146$

LEC 21 – Definite Integrals.

1. $\int_0^{\pi} \sin(x)\, dx$ Ans: 2

2. $\int_0^{\sqrt{\pi}} 3t \sin(t^2)\, dt$ Ans: 3

3. $\int_0^{\pi} t \sin(t)\, dt$ Ans: π