Physics: Waves and Fields: Notes

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Introduction

This document contains my notes from Physics: Waves and Fields.

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1 Chapter 15: Oscillatory Motion

1.1 Simple Harmonic Motion

While an object is oscillating, it's acceleration is no longer constant. We can however still understand its motion. Assuming the only force affecting the motion of an object is a spring, we can state that:

$$\sum \vec{F} = \vec{F}_{spring}$$

$$\sum \vec{F} = -k\vec{x}$$

...following this we can say:

$$-k\vec{x} = m\vec{a}$$

$$\frac{-k\vec{x}}{m} = \vec{a}$$

...remember that $\vec{a}$ is just the second derivative of position with respect to time:

$$\frac{-k\vec{x}}{m} = \frac{d^2\vec{x}}{dt^2}$$

...finally remember x is a function of time:

$$\frac{-k\ x(t)}{m} = \frac{d^2x(t)}{dt^2}$$

This equation is true for any object undergoing oscillatory motion, with only the spring force affecting it's motion. This is an example of a differential equation, as in an equation that contains both the function $x(t)$ and it's derivative (in this case second derivative). The goal is to find a function $x(t)$ that satisfies this equation. Differential equations are notably difficult to solve, and so a solution is given as the following:

$$\boxed{x(t) = Acos(\omega t + \phi)}$$

Where:

- A represents the amplitude of the oscillation.
- ω represents the angular frequency and is defined as:

$$\omega \equiv \sqrt{\frac{k}{m}}$$

- ϕ represents the phase constant, and can be found by:

$$\boxed{\phi = cos^{-1}\left(\frac{x(0)}{A}\right)}$$

Some other equations that are a result of this relationship are as follows:

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$$

$$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

$$\frac{\phi}{2\pi} = \frac{\Delta t}{T}$$

Where T represents period and f represents frequency.

Given by the fact that we have a function for the position of the oscillating mass, we can also find the functions for velocity, and acceleration as follows:

$$x(t) = Acos(\omega t + \phi)$$

$$v(t) = -A\omega sin(\omega t + \phi)$$

$$a(t) = -A\omega^2 cos(\omega t + \phi)$$

An interesting result of this is that we can find the values for the maximum *velocity* and *acceleration* because of the fact that they are sinusoidal functions and their max's are the absolute value of the leading coefficient. This means:

$$\boxed{v_{max} = A\omega = A\sqrt{\frac{k}{m}}}$$

$$\boxed{a_{max} = A\omega^2 = A\frac{k}{m}}$$

1.2 Energy in Simple Harmonic Motion

Through the course of the oscillation, total mechanical energy is converted from potential energy to kinetic energy, where potential energy is greatest at the end points, and kinetic energy is greatest in the middle. Consider the equation for the potential energy of a spring:

$$PE = \frac{1}{2}kx^2$$

The total energy is conserved, and so we can calculate the total energy at the end points (where there is no kinetic energy), as in:

$$\boxed{E_{total} = \frac{1}{2}kA^2}$$

Now that we know this, we can derive a formula for $v(x)$, using the fact that:

$$\Delta K + \Delta U = E_{total}$$

$$\frac{1}{2}kx^2 + \frac{1}{2}mv^2 = \frac{1}{2}kA^2$$

$$kx^2 + mv^2 = kA^2$$

$$mv^2 = kA^2 - kx^2$$

$$mv^2 = k(A^2 - x^2)$$

$$v^2 = \frac{k}{m}(A^2 - x^2)$$

$$\boxed{v(x) = \sqrt{\frac{k}{m}(A^2 - x^2)}}$$

1.3 Pendulums

Let's now investigate a specific form of a simple harmonic oscillator, a swinging pendulum. Consider, a point mass string up to the ceiling, as it swings it makes and angle θ with it's center vertical line. The length of the string is l. The point mass has two forces acting on it: gravity and the force of tension in the string. The force of gravity is the only force that causes torque, as in:

$$\sum \vec{\tau} = I\vec{\alpha}$$

$$-mglsin(\theta) = ml^2\vec{\alpha}$$

$$-gsin(\theta) = l\frac{d^2\theta}{dt^2}$$

$$-\frac{g}{l}sin(\theta) = \frac{d^2\theta}{dt^2}$$

Which for small angles is equivalent to:

$$-\frac{g}{l}\theta = \frac{d^2\theta}{dt^2}$$

Which is a differential equation we know how to solve:

$$\boxed{\theta(t) = \theta_{max}cos(\omega t + \phi)}$$

$$\text{Where: } \omega = \sqrt{\frac{g}{l}}$$

In the case where the pendulum is not a point mass, the only difference is that

$$I \neq ml^2$$

and so:

$$\omega = \sqrt{\frac{mgd}{I}}$$

1.4 Damped Harmonic Motion

In a realistic situation, the oscillatory motion will eventually stop. This is due to the friction within the spring, and interactions with the air molecules. This reduces the mechanical energy of the system. When the system is not ideal, the amplitude becomes a function of time, and the new solution to our differential equation must take that into account, as in:

$$x(t) = A(t)\ cos(\omega t + \phi)$$

... and in this case, A(t) can be written as:

$$\boxed{A(t) = A_i e^{-bt/2m}}$$

Where:

1. A_i is the natural amplitude of the oscillator without dampening, as in the initial amplitude.

2. b is the dampening parameter that encompasses the friction applied to the system.

3. ω is not the same ω as before, in this case it described as:

$$\omega = \sqrt{\omega_o^2 - \left(\frac{b}{2m}\right)^2}$$

...and here, ω_o is the same omega as before, as in the natural frequency of the oscillation without dampening.

1.5 Forced Oscillations

Just as friction does negative work on the system, it is possible to have a force acting on the system that does positive work. For instance, pushing someone on a swing, if you time your pushes properly then you can keep the person swinging. This is an important feature of forces acting on oscillations, that is that the force itself is periodic, and can be described as:

$$\boxed{F(t) = F_o sin(\omega t)}$$

where F_o is the driving force and ω is the angular frequency of the force.

In this situation, the position of an object can be written once again as:

$$x(t) = A(\omega)\ cos(\omega t + \phi)$$

... and in this case, A(ω) can be written as:

$$\boxed{A(\omega) = \frac{F_o/m}{\sqrt{(\omega^2 - \omega_o^2)^2 + (\frac{b\omega}{m})^2}}}$$

Where:

1. A(ω) is the amplitude when the driving force is applied.

2. F_o is the value of the driving force.

3. ω_o is the natural angular frequency as in: $\omega_o = \sqrt{\frac{k}{m}}$

4. ω is the frequency of the driving force.

2 Chapter 16: Wave Motion

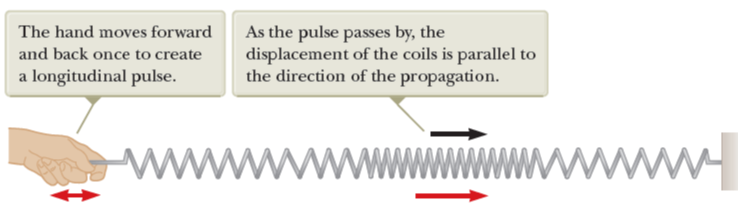
2.1 Wave Properties

For this class, a wave required three things:

1. Some source of disturbance.
2. A medium containing elements that can be disturbed.
3. Some physical mechanism through which elements of the medium can influence each other.

There are then two distinct types of waves:

1. **Transverse**: the elements of the medium that are in motion due to the wave are moving perpendicular to the direction of propagation.
2. **Longitudinal**: the elements of the medium that are in motion due to the wave are moving parallel to the direction of propagation.

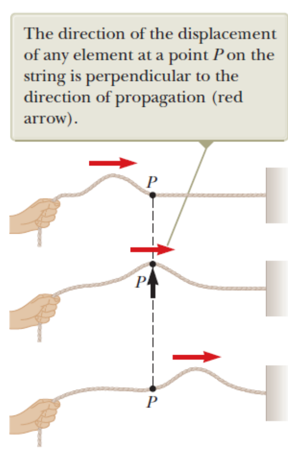


In order to mathematically describe a wave, we must define a multivariable function $y(x,t)$ which returns the y-value of a point of the disturbed medium at a specific point down the medium (an x-value, like a point down the rope) and at a specific time.

For a fixed $x_0 \rightarrow y(x_0,t)$ is a graph the motion of one particle through time. While for a fixed $t_0 \rightarrow y(x,t_0)$ is like a snapshot of the entire wave at a fixed time.

In general, for a wave which has the shape $f(x)$, the function describing the motion of the wave is

$$y(x,t) = f(x - vt)$$



Wavelength (λ) is the distance in meters between the crests of the wave. **Period** (T) is the time after which a full cycle completes. In particular, we are typically interested in studying sinusoidal waves in the form:

$$\boxed{y(x,t) = Asin(kx - \omega t + \phi)}$$

Where:

- $$k = \frac{2\pi}{\lambda}$$
- $$\omega = \frac{2\pi}{T} = 2\pi f$$
- $$v = \frac{\lambda}{T} = \lambda f$$

The speed v of a wave is a constant depending on the medium it moves through.

2.2 Waves Through a Rope

For a wave through a rope, the speed of the wave v is determined by the following equation:

$$\boxed{v = \sqrt{\frac{T}{\mu}}}$$

Where:

- T is the tension in the rope.
- μ is the mass per length of rope (like rope density).

The energy carried by a wave in this form is described by the following equation:

$$\boxed{E = \frac{1}{2}\mu\omega^2 A^2 \lambda}$$

And the power which is the energy per period of motion is described by:

$$\boxed{P = \frac{1}{2}\mu\omega^2 A^2 v}$$

2.3 Wave Reflection

For a wave travelling through a medium and hitting a boundary, some of the wave will be reflected back, and some will be transmitted through the boundary. Depending on the relative speeds of the wave on each side of the boundary, the reflected pulse may be inverted. In particular, if the speed of the wave is slower on the other side of the boundary, the reflected wave back into the original side of the boundary will be inverted (reflected across the x-axis).

2.4 Sound Waves

Sound waves are longitudinal waves which result from the compression or rarefaction (opposite of compression) of a medium.

In the context of sound waves, the mathematical qualities of the wave have physical effects. The frequency (f) is the perceived pitch or tone of the sound. For example the standard note A on a piano vibrates at a frequency of 440Hz (or 443Hz if you want to have an argument). Following this the amplitude (A) is the perceived volume of the sound.

For a longitudinal wave going through a medium, you can describe the position of any one element of the medium relative to its equilibrium position (a quantity called s) using the following function:

$$\boxed{s(x,t) = s_{max} cos(kx - \omega t + \phi)}$$

The wave can also be described by pressure differences of the particles in the medium compared to the equilibrium pressure in the following equation:

$$\boxed{\Delta P(x,t) = \Delta P_{max} sin(kx - \omega t + \phi)}$$

Notice $s(x,t)$ and $\Delta P(x,t)$ are out of phase by $\frac{\pi}{2}$.

2.5 Bulk Modulus

The bulk modulus (B) is a measure of how in-compressible a fluid is, a low B value means the fluid is easily compressible. The bulk modulus can be described by the equation:

$$\boxed{\Delta P = -B \frac{\Delta V}{V_i}}$$

Or equivalently:

$$\Delta V = -\frac{\Delta P \; V_i}{B}$$

From the first equation it can be concluded that:

$$\boxed{\Delta P_{max} = B s_{max} k = s_{max} \rho v \omega}$$

Recall that on a string, the speed of the wave can be described by: $v = \sqrt{T/\mu}$, and now for a longitudinal wave that is more generally:

$$v = \sqrt{\frac{\text{elastic property}}{\text{inertial property}}} = \sqrt{\frac{B}{\rho}}$$

The speed of sound through air can be approximately calculated as a function of the temperature using the following formula:

$$v = 331m/s\sqrt{1 + \frac{T}{273°C}}$$

2.6 Sound Intensity and Loudness

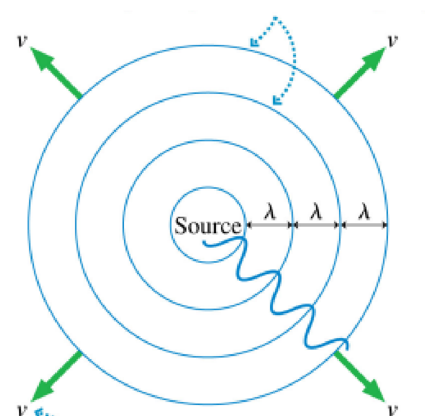
Most waves travel in more than 1 dimension and so they spread uniformly from a point. The **wave fronts** are the maximums of the wave, expanding from a point and are always spaced one wavelength apart.

For a sound wave the energy contained in the wave can be described by the following equation, which is similar to that of the energy in a string:

$$E = \frac{1}{2}\rho\omega^2 s_{max}^2 \lambda \times (\text{Area})$$

Where the area in the formula is the area of the surface causing the longitudinal wave motion (like the vibrating surface of a speaker). The power is:

$$P = \frac{1}{2}\rho\omega^2 s_{max}^2 v \times (\text{Area})$$



Energy is always conserved, if you had a detector in the shape of a sphere around the source, then it would pick up 100% of the energy. The fraction of energy you detect is the fraction of the area of a sphere your detector picks up. As in:

$$(\text{power})_{\text{detected}} = (\text{power})_{\text{emitted}} \times \frac{(\text{Detector Area})}{4\pi r^2}$$

If you want to consider the intensity of the sound wave felt at different distances from the source, the following equations help:

$$I = \frac{\text{Power}}{\text{Area}} = \frac{1}{2}\rho\omega^2 v s_{max}^2$$

$$I = \frac{\text{Power}}{4\pi r^2}$$

The sound level of a sound wave is measured in decibels (which is a dimensionless quantity), and can be calculated by:

$$\boxed{\beta = 10\ log_{10}(\frac{I}{I_0})}$$

Where:

- β is the sound level in decibels.
- I is the intensity of sound at the observer.
- I_0 is the threshold of hearing which is a constant equal to $10^{-12}\ W/m^2$

Decibels (β) are not additive, but the intensities are.

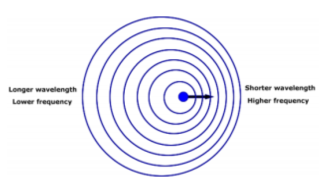
2.7 The Doppler Effect

For this entire section we assume that the medium is at rest. The frequency measured by an observer may differ from the emitted frequency if the source or observer are moving. This is an effect that applies to all waves. What happens is the motion changes the frequency at which the observer crosses the wave fronts. Here are the two possibilities for a moving observer and motionless source:

1. If moving toward the source, they measure a higher frequency.
2. If moving away from the source, they measure a lower frequency.

Here are the two possibilities for a moving source and motionless observer:

1. If moving towards the observer, the higher the frequency observed.
2. If moving away from the observer, the lower the frequency observed.



A useful equation relating to the Doppler effect is the following:

$$\boxed{t = \frac{L}{v_o + v}}$$

Where:

- t is the time it takes for the first wave front to encounter the observer.
- L is the initial distance between the observer and the source.
- v_o is the velocity of the observer.
- v is the speed of the wave in the medium.

The main equation relating to the Doppler effect finds the frequency observed by the observer (f_o) relating it to the frequency emitted by the source (f_s):

$$\boxed{f_o = f_s \left(\frac{v + v_o}{v - v_s} \right)}$$

One thing to note is the sign on the variables v_o and v_s. The sign always reflects whether the person is moving towards or away from the other. v always has a positive sign.

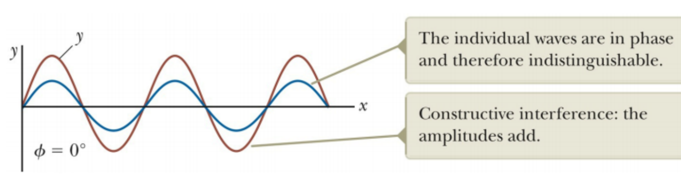
3 Chapter 17: Superposition and Standing Waves

3.1 Interference

The **superposition principle** states that when two waves collide they add together point by point, and pass through each other unaffected.

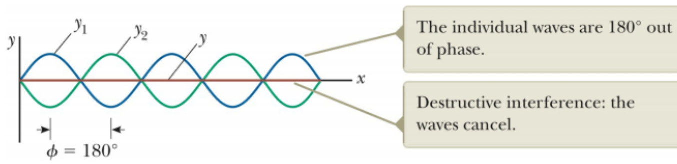
In this course, we consider the collision of two waves with the same amplitudes and different frequencies and phase.

First consider two waves each with the same amplitude, frequency and phase shift, essentially the same wave. If both these waves were to propagate through a medium at the same time, the effect would be the same as having a single wave with twice the amplitude. This is because at every point the the amplitude of the resulting wave is equal to the sum of the two heights of the individual waves. As visible in the following image, the blue curve represents both of the individual waves overlapped over each other, and the red wave represents the resulting wave:

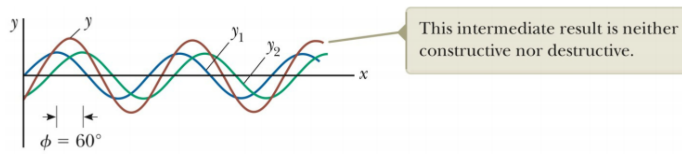


This is an example of *constructive interference* which occurs throughout the wave where the waves are both adding onto the total amplitude.

The following is an example of *destructive interference*. The blue and green waves are identical except out of phase by half a period, and the red line is the resulting curve:



Typically however, waves are neither perfectly in phase nor perfectly out of phase by half a period. In this case, the resulting amplitude is somewhere in between 0 and twice the amplitude of the two waves. An example of this is in the following image:



You can determine the phase difference ($\Delta\phi$) between two waves emitting from a source depending on the distance the wave travels from each source to the observer:

$$\boxed{\Delta\phi = 2\pi \left(\frac{|r_2 - r_1|}{\lambda}\right)}$$

The result of adding the two sinusoidal waves:

$$y(x,t) = Asin(kx - \omega t + \phi) \text{ and } y(x,t) = Asin(kx - \omega t)$$

... which have the same amplitude and frequency is (using trigonometric identities):

$$\boxed{y(x,t) = 2Acos\left(\frac{\phi}{2}\right) sin\left(kx - \omega t + \frac{\phi}{2}\right)}$$

Assuming two waves leave their individual sources in phase, and travel different distances (r_1 and r_2) to an observer, then:

- Constructive interference occurs if:

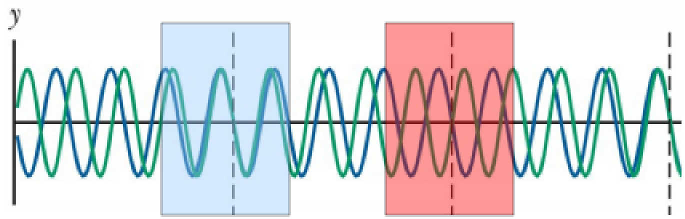
$$|r_2 - r_1| = n\lambda, \text{ for } n \in \mathbb{Z}^+$$

- Destructive interference occurs if:

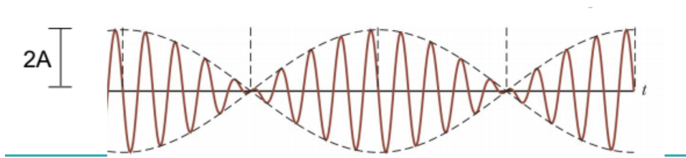
$$|r_2 - r_1| = (n + \frac{1}{2})\lambda, \text{ for } n \in \mathbb{Z}^+$$

3.2 Beat Frequency

Now consider two waves that have the same amplitude but different frequencies. When these waves are added together, we get a mixture of zones with constructive interference, and zones with destructive interference. As visible in the following image.



The sum of these waves has a unique shape, essentially it is a high frequency sinusoidal wave that is being *enveloped* by a relatively slower sinusoidal wave, as visible in the following image:



Those dips in the wave, as in moments of complete destructive interference, are called **beats**. Similar to how with damped oscillations in *chapter 15*, the amplitude became a function of time, the amplitude is now also a function of time. We can describe a wave like this by:

$$y(t) = A\cos(\omega_1 t) + A\cos(\omega_2 t)$$

$$\boxed{y(t) = 2A\cos\left(\frac{\omega_2 - \omega_1}{2}t\right)\,\sin\left(\frac{\omega_1 + \omega_2}{2}t\right)}$$

Notice:

$$A(t) = 2Acos\left(\frac{\omega_2 - \omega_1}{2}t\right)$$

Also notice that since ω is a positive value, the frequency of the *sin* function is greater than that of the *cos* function. From that $A(t)$ function, we can derive a formula for the **beat frequency** of the sinusoidal function, as in the frequency of the points of constructive/destructive interference, as below:

$$f_{\text{beat}} = |f_2 - f_1|$$

... which is that the absolute difference between the frequencies of the original two sinusoidal waves is the beat frequency. Notice also by looking at the graph that the *beat frequency* is twice the envelope frequency.

3.3 Standing Waves

A standing wave is a wave created by the sum of two waves. Those two waves have the same amplitude and frequency, but are travelling in opposite directions. Standing waves have no horizontal motion as time moves forward. A unique feature of standing waves is that certain points called **nodes** experience no movement at all, while other points called **antinodes** experience the most movement to the full amplitude of 2A.

A standing wave which is the sum of two sinusoidal waves with equal amplitudes and frequencies moving in opposite directions can be modeled by the following equation:

$$\boxed{y(x,t) = [2Asin(kx)]cos(\omega t)}$$

Here, $[2Asin(kx)]$ acts as the amplitude as a function of position, which is notably constant with time. Therefore each individual particle in the wave is undergoing simple harmonic motion with the same frequency and amplitude determined by that expression.

Finding Nodes/Antinodes

Nodes occur where the value of $[2Asin(kx)]$ is zero, as in:

$$kx = n\pi, \text{ for } n \in \mathbb{Z}$$

... or equivalently, to find the position of the n^{th} node (x_n):

$$x_n = \frac{n\lambda}{2}, \text{ for } n \in \mathbb{Z}$$

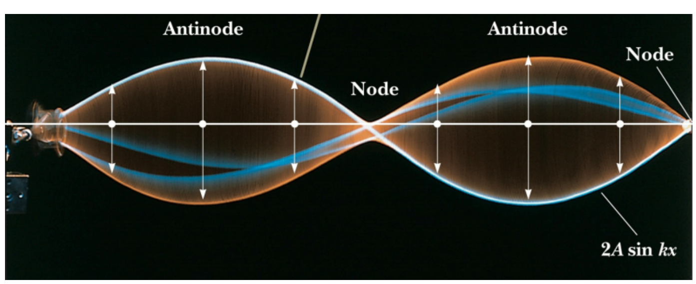
Antinodes occurs where the value of $[2Asin(kx)]$ is 2A (the maximum amplitude), as in:

$$kx = \frac{m\pi}{2}, \text{ where } m \text{ is an odd integer}$$

... or equivalently, to find the position of the m^{th} antinode (x_m):

$$x_m = \frac{m\lambda}{4}, \text{ where } m \text{ is an odd integer}$$

The distance between antinodes is *half* a wavelength (λ), the distance between nodes is half a wavelength, and the distance between a node and the nearest antinode is one quarter of a wavelength.



3.4 Standing Waves on a String

Recall that if a wave is sent down a string and it hits a fixed end, it will be reflected back. If both ends of a string are fixed in place, and a wave is passed through the string, the ends must be nodes. Since nodes are always half a wavelength apart, standing waves are only created if the distance between each end point is a multiple of $\frac{\lambda}{2}$, as in:

$$\boxed{L = \frac{n\lambda}{2}}, \text{ for } n \in \mathbb{Z}$$

... or equivalently:

$$\boxed{\lambda_n = \frac{2L}{n}}, \text{ for } n \in \mathbb{Z}$$

Here n denotes how many multiples of half a wavelength are between the two end points. This is called the *harmonic* of the standing wave and the n^{th} harmonic has n antinodes. The first harmonic is called the *fundamental frequency* or just the *fundamental*.

The oscillating frequency that achieves this n^{th} harmonic can be found by:

$$\boxed{f_n = \frac{nv}{2L}}, \text{ for } n \in \mathbb{Z}$$

The fundamental frequency is the note heard by playing string instruments, this note can be changed by changing the length of the string (putting your finger somewhere along the string) or by changing the tension in the string (tuning the instrument) which changes the speed of the wave.

3.5 Standing Waves in Air Columns

In a pipe, a sound wave can travel through and either be reflected back by a closed end or just exit the pipe by an open end. As a sound wave travels down a pipe, a closed end *must be a node* since the molecules cannot oscillate at the end. Likewise, an open end *must be an antinode*.

Consider a pipe with both ends open, since both ends are antinodes, and since antinodes are always half a wavelength apart, then to achieve a standing wave the length of the pipe must be a multiple of $\frac{\lambda}{2}$. This is identical to the standing wave on a string except with antinodes instead of nodes. Mathematically that means that:

$$\boxed{L = \frac{n\lambda}{2}}, \text{ for } n \in \mathbb{Z}$$

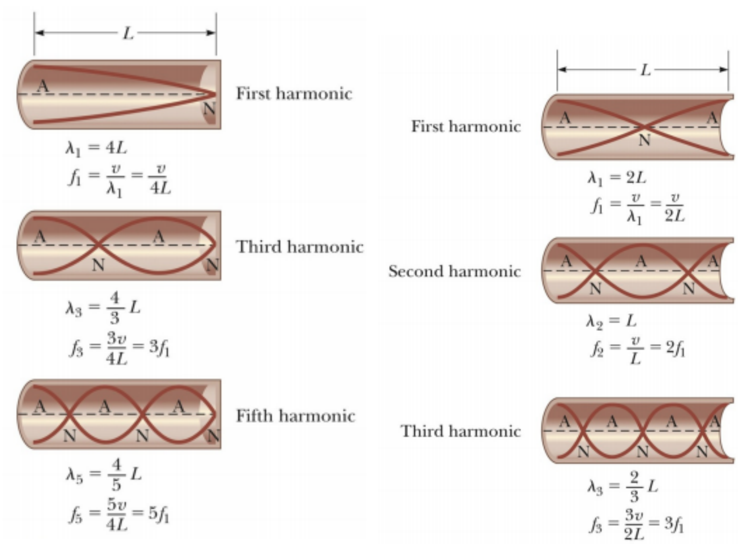
$$\boxed{f_n = \frac{nv}{2L}}, \text{ for } n \in \mathbb{Z}$$

Now **consider when a pipe is open only on one end**. Since the open end must be an antinode and the closed end must be a node, and since nodes and antinodes are always $\frac{1}{4}$ of a wavelength apart, then to achieve a standing wave, the length of the pipe must be an odd multiple of $\frac{\lambda}{4}$, as in:

$$\boxed{L = \frac{m\lambda}{4}}, \text{ where } m \text{ is an odd integer}$$

$$\boxed{f_m = \frac{mv}{4L}}, \text{ where } m \text{ is an odd integer}$$

Notice, there are no even harmonics for a pipe with once end closed.

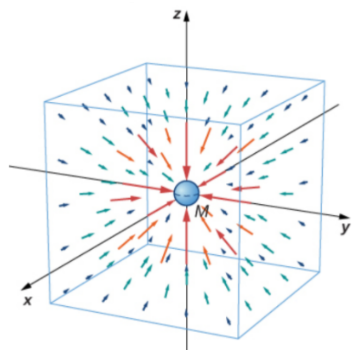


4 Chapter 13: Universal Gravitation

4.1 Newtonian Gravity

Gravity is a **vector field**. A field is a function that attributed a value to every point in space. For example, temperature is an example of a **scalar field**, in which every point in the room has a temperature. A *vector field* is a function which assigns a vector to every point in space.

The **gravitational field** is the field which assigns a vector to each point in space around a mass. The *gravitational field* from a single point mass always points towards the center of the mass. **Mass** creates a *gravitational field*. Masses react to each other's *gravitational fields*.



The magnitude of any vector in the *gravitational field* is defined by:

$$\boxed{|\vec{g}| = \frac{GM}{r^2}}$$

Where:

- G is a constant defined by

$$G \equiv 6.67 \times 10^{-11} \frac{m^3}{kg\ sec^2}$$

- M is the mass of the object creating the field.
- r is how far away the point is from the center of the massive object.

Near the earth's surface, the gravitational field essentially points straight down, and we can use the formula above to show that near the surface:

$$|\vec{g}| = 9.81 \frac{m}{sec^2}$$

If a mass is affected by multiple *gravitational fields*, then you just take the vector sum of each gravitational field to find the resultant vector.

A mass m sitting in a field $\vec{g}$ feels a force:

$$\boxed{\vec{F} = m\vec{g}_{ext}}$$

All particles in the universe exert gravitational forces on each other, and the magnitude of the force between two masses m_1 and m_2 is described by *Newton's Law of Universal Gravitation*:

$$\boxed{\left|\vec{F}_G\right| = \frac{Gm_1m_2}{r^2}}$$

This force is always attractive and points in a line connecting the centers of mass of the two objects.

4.2 Orbits and Gravitational Potential Energy

Using Newton's second law, and laws of circular orbits, we can see derive a formula for the velocity v of a mass M_p as it orbits at a radius r around a mass M_s, as in:

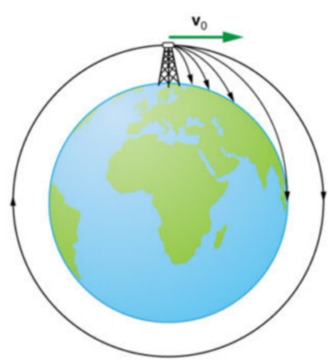
$$\sum \vec{F} = m\vec{a}$$

$$\frac{GM_sM_p}{r^2} = M_p\frac{v^2}{r}$$

... finally:

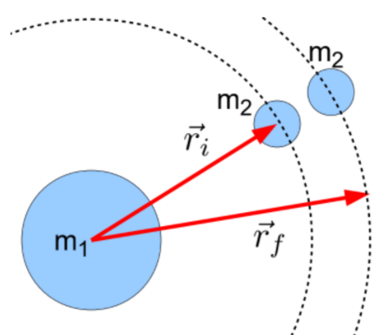
$$\boxed{v^2 = \frac{GM_s}{r}}$$

Orbit is essentially a free fall in which the Earth continuously recedes from under the mass falling.



Geostationary orbit is the orbit which has the same period as the rotation of the Earth, for this reason a mass orbiting the sky in *geostationary orbit* would remain in the same place in the sky. The distance above the Earth's surface for *geostationary orbit* is around $36,000km$.

If you were to imagine pulling a mass from an initial radius to a final one, what would be the work done by gravity? As in the following image:



Using the fact that:

$$W_{gravity} = \int_{r_i}^{r_f} \vec{F} \cdot d\vec{r}$$

... and:

$$\Delta U_g = -W_{gravity}$$

$$\boxed{\Delta U_g = \frac{Gm_1m_2}{r_i} - \frac{Gm_1m_2}{r_f}}$$

Which tells us that:

$$\boxed{U_g = -\frac{Gm_1m_2}{r}}$$

Where U_g is the potential energy of a pair of masses, and r is the center to center distance between the objects. The zero of potential energy is typically set at $r = \infty$, and so separating objects takes positive work. The potential energy is *stored* between the pair of the objects, not just one of the objects.

The **total energy in a circular orbit** is defined as:

$$\boxed{E = \frac{-GMm}{2r}}$$

Where:

- M is the mass of the large object.
- m is the mass of the small object orbiting.

The **total energy in an elliptical orbit** is given by:

$$\boxed{E = -\frac{GMm}{2a}}$$

Where a is the *semi-major* axis of the ellipse (the long axis).

4.3 Escape Speed

Escape speed is the speed a projectile needs to be launched at in order to escape the gravitational field and go off forever. The final radius of an object that has been launched is given by:

$$\boxed{r_f = \frac{GM}{\frac{GM}{r_i} - \frac{1}{2}v_i^2}}$$

Notice that if:

$$\frac{1}{2}v_i^2 \geq \frac{GM}{r_i}$$

... then the object never stops. You can solve for v_i to find the launch speed required for the object to escape the gravitational field, as in:

$$\boxed{v_{esc} = \sqrt{\frac{2GM}{R}}}$$

5 Chapter 22: Electric Fields

5.1 Elementary Concepts

Electric fields behave very similarly to gravitational fields. Just like how a GF is created by a mass in space, an electric field is created by a *charge* in space. The main difference between mass m and charge q or Q is that charge can take on a positive or negative value. This is the main important distinction between gravitational fields and electric fields, *there are two types of charges, positive and negative.* The **electric force** caused by the electric field is the force that acts between electrically charged objects.

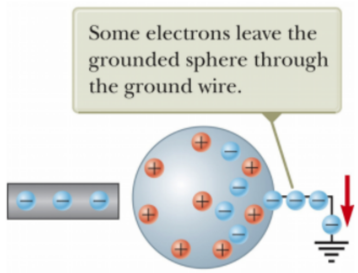
Another important fact about electric charges is that the net electric charge of the universe is conserved. This means that if I have an object with a charge of $+2$, the somewhere in the universe there **must** be an object with charge -2.

The following are the elementary concepts regarding electric forces:

- Like charges repel, opposites attract.
- Electric forces get weaker as the distance between charges increases.
- Only negative charges move, positive charges stay in place.
- Neutral objects have an equal number of positive charges and negative charges.
- A positively charged object has a *net positive charge.*
- A negatively charged object has a *net negative charge.*
- An *insulator* is a material which resists the flow of charges.
- An *conductor* is a material which promotes the flow of charges.
  - Note: Insulators can be polarized by moving all the positive charges to once end and thus the negative charges to the other. This **does not** allow charges to move out of the insulator. This is called *charging by induction.*

5.2 Charging by Induction

A neutral object can be charged by induction by bringing a negative object close to it with a ground wire on the other side. The negative charges inside the neutral object will be repelled into the ground wire giving the object an overall positive charge. The following image sums up this concept:



The *SI Unit* or charge is the **coulomb** (C), which is a huge charge. Every physical unit of charge is actually a multiple of the *fundamental unit of charge* (e) which is defined as:

$$\boxed{e = 1.602 \times 10^{-19} C}$$

The charge of a proton (q_p) and the charge of an electron (q_e) are:

$$q_p = +e$$

$$q_e = -e$$

An **electric field** is the distortion of space due to an electric charge. The electric field $\vec{E}$ always points *away from* a *positive charge* and *towards* a *negative charge.* The force felt by a charge in an electric field is:

$$\boxed{\vec{F_e} = q\vec{E}}$$

... where:

$$\boxed{\left|\vec{E}\right| = \frac{k_e|q|}{r^2}}$$

For an electric field created by a point charge. k_e is the proportionality constant with a value of $9.0 \times 10^9 \frac{N \cdot m^2}{C^2}$.

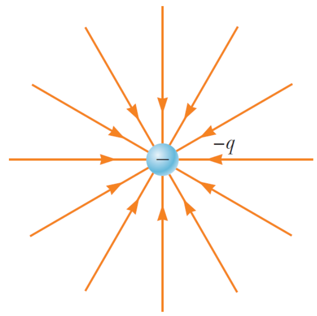
In the presence of multiple point charges, the net field is found by adding the fields from each point charge separately, as vectors.

Coulomb's Law states that the magnitude of force exerted on one point charge onto another point charge at a distance is:

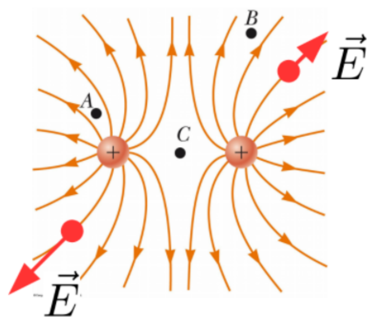
$$\left|F_{1on2}^{\rightarrow}\right| = \left|F_{2on1}^{\rightarrow}\right| = \boxed{\left|\vec{F}\right| = \frac{k_e|q_1 q_2|}{r^2}}$$

5.3 Electric Field Lines

Another way to visualize an electric field around a charge is drawing **Electric Field Lines**. These are lines that come from the charge, and for every point on the line, the electric field is tangent to it. In general, the density of lines represent the strength of the field. These field lines can be curved, however they cannot cross and only end at a charge or at infinity. For a simple single point charge, the following is the field line representation of it's electric field:



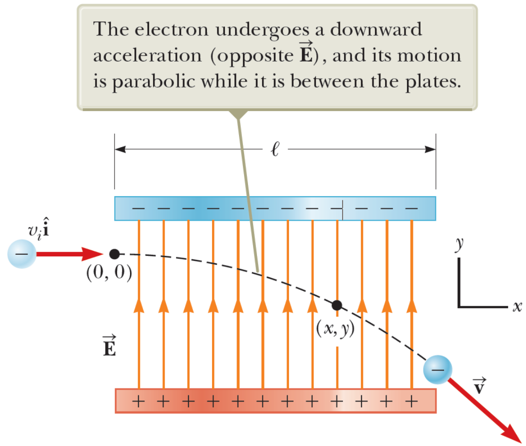
... and the following is the field line representation for a system of two charges:



You can see that the field lines curve, this does **not** mean that the electric field curves. The electric field is a vector and it always tangent to the field line. Another important note is that these field lines are surrounding the object in 3D, and that the electric field still exists between field lines. The direction of the line is that of the field on a positive charge placed in the field.

5.4 Uniform Electric Field

A uniform electric field is created by two charged plates that produce an electric field that essentially goes directly between the plates, and always perpendicular to them. The following image shows this:



Notice that a single particle travelling through a uniform electric field is experiencing *constant acceleration*. Which means that the kinematics of constant acceleration can be used to describe the motion of the particle.

$$\boxed{|\vec{a}| = \frac{|q|\left|\vec{E}\right|}{m}}$$

6 Chapter 23: Electric Flux and Gauss's Law

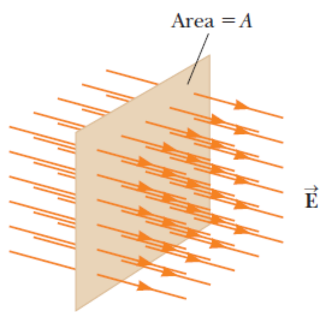
6.1 Electric Flux

Electric flux is another measure of charges that related to the *field lines* moving through a surface. Given some finite flat area A, the **electric flux** Φ_E through that flat surface is defined by:

$$\Phi_E = \vec{E} \cdot \vec{A}$$

Where:

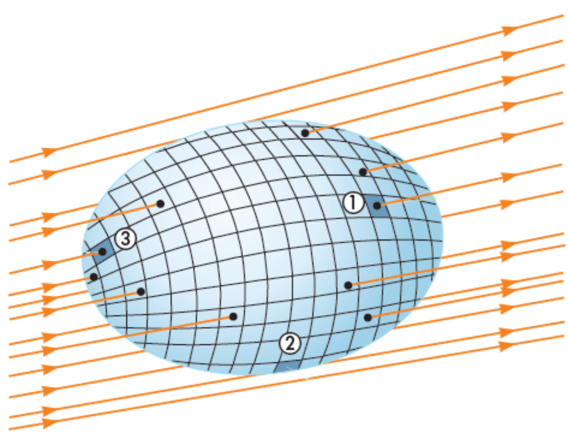
- $\vec{E}$ is the electric field passing through the surface.
- $\vec{A}$ is the vector which is normal to the plane which defines the finite flat area.



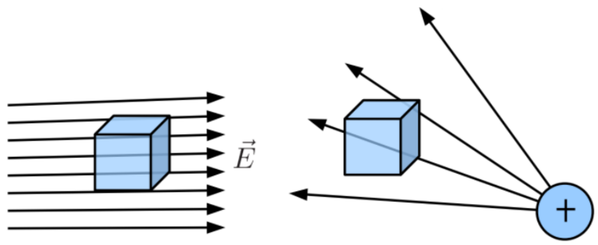
A **closed surface** is a surface with volume, as in it encloses some space. Flux is defined to be positive if it is *exiting* the surface, and negative if it is *entering* the surface. In general, to find the flux through a surface, you divide up the surface into little pieces, compute the flux through each piece, and then integrate them back together. Doing this over a surface is called a surface integral and is written as the following:

$$\Phi_E = \oint \vec{E} \cdot d\vec{A}$$

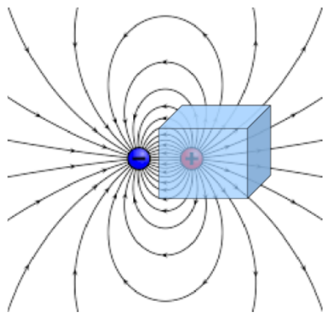
... which is the definition for electric flux through a closed surface, visualized by the following:



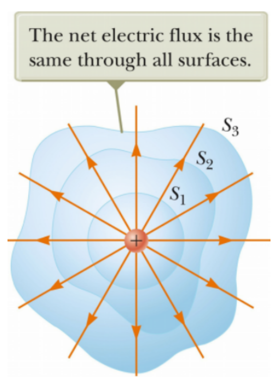
The **net flux** through a surface is an interesting thing to think about, in the following two examples the net flux is **zero**:



... this is because the amount of field lines entering equals the number of field lines leaving, and so the net flux is zero. However in the following example, the next flux is not zero:



... since field lines are only leaving the box. Notably in order to have a net flux not equal to zero, you need to have a net charge within the volume enclosed by the surface. The net flux does not depend on the shape of the surface, all of the following surfaces have the same field lines going through them:



6.2 Gauss' Law

The net flux through a closed surface is non-zero if there is a net charge inside the surface. **Gauss' Law** states that:

$$\Phi_E^{net} = 4\pi k_e Q_{in}$$

This means that:

$$\oint \vec{E} \cdot d\vec{A} = 4\pi_e Q_{in}$$

... where Q_{in} is the net charge on the inside of the surface. This gives us a way to calculate the electric flux through a surface without having to compute the surface integral. This is also commonly written as:

$$\boxed{\oint \vec{E} \cdot d\vec{A} = \frac{Q_{in}}{\varepsilon_0} = \Phi_E^{net}}$$

... where ε_0 is called the permitivity of free space defined as:

$$\varepsilon_0 = \frac{1}{4\pi k_e} = 8.85 \times 10^{-12} \frac{C^2}{N \cdot m^2}$$

Gauss's Law is true for any closed surface. A *Gaussian Surface* is a fictitious surface made up to solve a problem.

Gauss's Law confirms some facts that we already knew, like that from the outside of a point charge, the electric field caused by that point charge is:

$$\left|\vec{E}\right| = \frac{k_e Q}{r^2}$$

... but Gauss' Law also tells us that the electric field caused by a point charge, within its own radius is as follows:

$$\boxed{\left|\vec{E}\right| = \frac{k_e Q}{a^3} r}$$

Where:

- a is the radius of the entire point charge.
- r is the distance from the center of the point charge.
- Q is the total charge of the point charge.

Note: $\lambda = Q/L$ is linear charge density, $\sigma = Q/A$ is surface charge density, and $\rho = Q/V$ is volume charge density.

Gauss' Law also can tell us about the electric field caused by an in finite plate of charge. The electric field is constant regardless of the distance from the charged plate, so long as that distance is much smaller than the size of the plate. The electric field can be described by:

$$\boxed{\left|\vec{E}\right| = \frac{Q}{2A\varepsilon_0} = \frac{\sigma}{2\varepsilon_0}}$$

Finally, Gauss' Law is helpful in finding the electric field caused by a long wire or charge, which can be described by:

$$\boxed{\left|\vec{E}\right| = \frac{2k_e \lambda}{r}}$$

7 Chapter 24: Electrical Potential

7.1 Electric Potential

For some position of a charge in an electric field, the charge-field system has a potential energy. This potential energy depends on both the charge creating the field, and the charge experiencing the field. Dividing this potential energy by the charge experiencing the electric field gives us a new quantity called **electric potential** (V) as follows:

$$V = \frac{U_E}{q} = \frac{J}{C} = Volt$$

... which is the potential energy per charge. The **potential difference** between two points in an electric field is defined as the change in electric potential energy of the system when a charge q moves between the two points divided by the charge:

$$\Delta V = \frac{\Delta U_E}{q} \implies \boxed{\Delta U_E = q\Delta V}$$

The potential difference ΔV exists solely because of a source charge, while difference in potential energy depends on both the source charge and the charge experiencing the field. The potential difference is describing how much potential energy a specific charge q would have at some point. Consider how height affects potential energy in the equation $\Delta U_g = mg\Delta h$. If I were to place some mass m at some point with height h, you could tell me the difference in potential energy it would have. Similarly, if I were to place some charge q at some point with voltage ΔV, you could also tell me the difference in potential energy it would have. In a uniform electric field, the potential difference can be described by:

$$\boxed{\Delta V = V_f - V_i = -\vec{E} \cdot \Delta\vec{s}}$$

In a non uniform electric field, the general form of the potential difference is given by:

$$\Delta V = \int_{state\ 1}^{state\ 2} \vec{E} \cdot d\vec{s}$$

An **electron volt** is a unit of energy defined as:

$$\boxed{eV \equiv (1.60218 \times 10^{-19} C)(1V) = 1.60218 \times 10^{-19} J}$$

For a point charge, the electric potential at any radius away from it's center is described by:

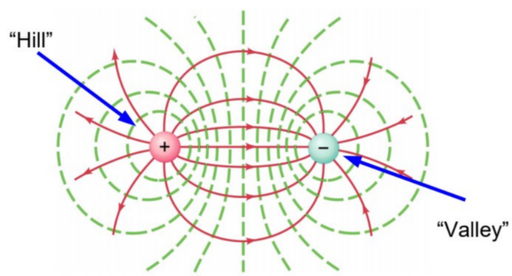
$$\boxed{V(r) = \frac{k_e Q}{r}}$$

... by doing this we are setting $V(\infty)$ to be our zero of electric potential. This creates radii of equal voltage, called equipotential surfaces that surround the point charge. A positive charge creates "hills" (as in a point of high potential that decreases with distance) while a negative charge creates "valleys" (as in a point of low potential that increases with distance). Comparing equipotential lines to the lines of a topographic map is a common comparison. The electric field $\vec{E}$ is like the slope of the hills.

Positive charges move or "roll downhill" to lower potential, while negative charges act in the opposite direction, they tend to "roll uphill" to a higher potential.

The Superposition Principle hold with potentials, so if you have n charges around a point then:

$$V_p = \sum_{n=1}^{n} V(r_n)$$



The potential energy for a pair of point charges is given by the following:

$$\boxed{U_E = \frac{k_e q_1 q_2}{r}}$$

... which is very similar to the potential energy of a pair of masses.

In order to find the electric field ($\vec{E}$) from the potential difference, you need to extract the three spacial components of the electric field (E_x, E_y, E_z) by:

$$E_x = -\frac{\partial V}{\partial x}$$

$$E_y = -\frac{\partial V}{\partial y}$$

$$E_z = -\frac{\partial V}{\partial z}$$

... or radially:

$$E_r = -\frac{\partial V}{\partial r}$$

7.2 Conductors

Conductors are materials that allow electrons to move freely inside them. **Electrostatics** is the study of charge distributions that do not change with time, this is what we have been doing. If an electric field acts on a conductor, the electrons inside of it will rearrange themselves to produce a sheet of negative charges, this sheet of negative charges produces its own electric field, which cancels out the external electric field. This takes times, and while the electric fields are not balanced, the resulting force is what moves the electrons to the equilibrium position. For this reason, the electric field ($\vec{E}$) inside of a conductor is always 0 at equilibrium.

There are **5 main properties of conductors** in equilibrium:

1. The electric field is zero everywhere inside.
2. Any net charge on the conductor will reside on its surface.
3. Near the surface, $\vec{E}$ points perpendicular to the surface and has magnitude:
$$\left|\vec{E}\right| = \frac{\sigma}{\varepsilon_0}$$
4. The charge density is highest where the radius of curvature of the surface is smaller.
5. The surface of a conductor is an equipotential.

8 Chapter 26: Current and Resistance

8.1 Electric Current, and Ohm's Law

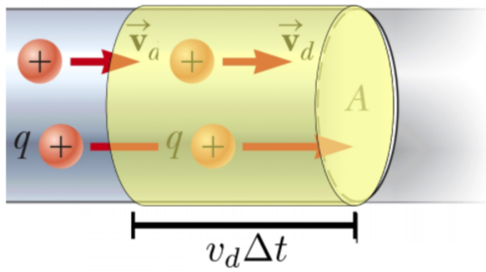
Electrons in a circuit do work but are not used up. Recall that potential difference is like a hill which can cause charges to move, caused by a force induced by an electric field. Applying a potential difference to a material with free charges (electrons) leads to a current which is the flow of charge, as in:

$$I = \frac{dQ}{dt}$$

$$1A = 1C/s$$

Note: Current is defined as the flow of *positive charges* even thought in reality it is the negative charges that are moving.

Say we have a wire like the following:



... and we want to know the number of charges N that flow past the circular area A in time Δt. Suppose we know the concentration of charges n, and the velocity of each charge v_d. Charges that are further than $v_d \Delta t$ meters away will not make it through the circle in Δt time so we only need to find the number of particles in that distance, which we can get by simply multiplying the charge density with that volume, as in:

$$N = n(A(v_d \Delta t)) = \boxed{nAv_d \Delta t}$$

Now say we want to know the total value of charge that passes through the area A in the same time, we just multiply the number of charges by the value of each charge, as in:

$$\boxed{\Delta Q = qN = qnAv_d \Delta t}$$

We can use this to define current as:

$$\boxed{I = \frac{\Delta Q}{\Delta t} = nqAv_d}$$

The velocity of the particles is called the **drift velocity** because in reality, the motion of the electrons are fast and random, changing directions and colliding, but an electric field induces a small drift velocity to all the electrons in the opposite direction than the field, and this is what's used.

Ohm's Law states that the voltage drop across a resistor, as in the potential difference between each side of the resistor, is proportional to the current passing through the resistor, as in:

$$\boxed{\Delta V_R = IR}$$

... where R is the resistance measure in Ohms (Ω) and the higher the resistivity (ρ) the more insulating the material is. Conductivity is positive, the higher the conductivity the more flow of charge for a given electric field, denoted (δ). Note that:

$$\rho = \frac{1}{\delta}$$

Conductivity and resistivity are properties of the material and temperature, and do not depend on size or shape. The resistance of a resistor is defined in terms of its resistivity:

$$\boxed{R = \rho \frac{\ell}{A}}$$

... where ℓ is the length of the resistor in the direction of the current, and A is the cross-sectional area.

Power is the rate at which charges lose potential energy as they pass through a resistor and is defined as:

$$\boxed{P = I\Delta V_R}$$

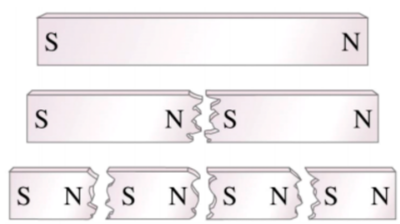
... where I is the current through the resistor, and ΔV_R is the voltage drop across the resistor. Combining this with *Ohm's Law* we can find more formulas for power:

$$P = I\Delta V_R = I^2 R = \frac{\Delta V_R^2}{R}$$

9 Chapter 28: Magnetic Fields

9.1 Magnetic Force and Field

Magnetism is a third long range force that acts through a field. The *charges* of a magnet are called poles, north and south. Like poles repel, opposites attract. A unique detail about magnetic fields and charges is that free magnetic charges don't exist, a magnetic pole is always counteracted by another magnetic pole of opposite polarity.



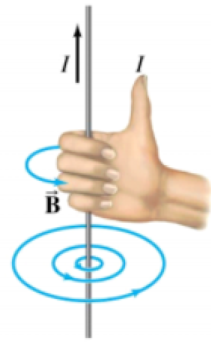
As with electric fields, there is a magnetic field vector at every point in space. Since there are no free charges, all magnetic field lines are closed loops. Magnetic fields $\vec{B}$ are defined in:

$$[\vec{B}] = \frac{kg}{C \cdot s} \equiv T(Tesla)$$

Currents produce magnetic fields.

In the specific case of current through a wire, the direction of the field is a circle around the wire. To determine whether the field is moving clockwise of counter-clockwise. You can use the first **Right-Hand Rule** which is:

- Thumb of right hand points in direction of current.
- Curl of fingers in direction the magnetic field.



9.2 Magnetic Force Law

The magnetic force induced on a charge by a magnetic field is:

$$\boxed{\vec{F}_B = q(\vec{v} \times \vec{B})}$$

... which is a vector with magnitude:

$$\left|\vec{F}_B\right| = |q||\vec{v}|\left|\vec{B}\right|\sin(\theta)$$

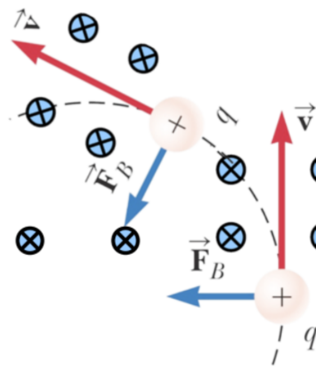
... and direction in the direction of the cross product if $q > 0$ or opposite if $q < 0$. Recall the cross product is a vector perpendicular to the plane containing $\vec{v}$ and $\vec{B}$. To determine direction, use the next **right hand rule**:

- Point your fingers on your right hand in the direction of the motion of the particles.
- Curl your fingers into the magnetic field.
- The direction your thumb is pointing is the direction of the force for a positive charge, opposite to the direction for a negative charge.

9.3 Motion in a Uniform Magnetic Field

Note that the following notation is used to denote the direction of the magnetic field into and out of the page:

Consider a particle moving through a uniform magnetic field, as in the following image:



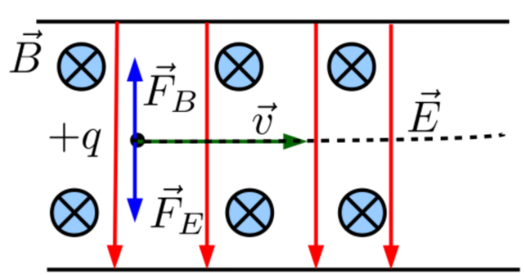
Since the velocity is in the positive y direction, and the magnetic field is going into the page, by the right hand rule the force will always point to the right. This causes a particle in a uniform magnetic field to move in a circular path, the radius of which is called the **cyclotron radius**, and can be calculated by:

$$\boxed{r = \frac{m|\vec{v}|}{|q|\left|\vec{B}\right|}}$$

An important thing to consider is that magnetic forces never point in the direction of velocity, always perpendicular to it, and so they cannot speed up or slow down particles. Magnetic fields do no work on charges particles. If the magnetic field is not perfectly perpendicular to the velocity of the particle, you just take the perpendicular component of the velocity. This results in a helical path.

9.4 Velocity Selector

A velocity selector is a device that allows you to choose the velocity of particles. This work by shooting a bunch of particles through the velocity selector, and only the ones with the correct velocity make it through. The inside of a velocity selector looks like the following:



The particles has two forces acting on it, one magnetic force pointing upwards, and one electric force pointing down. The magnetic force is dependant on velocity, and so if the particle moves too fast, then the magnetic force will be greater than the electric and the particle will go to the roof. Similarly if the particle's speed is too low the particle will crash into the floor. There is a specific velocity that need to be met for the following to be true:

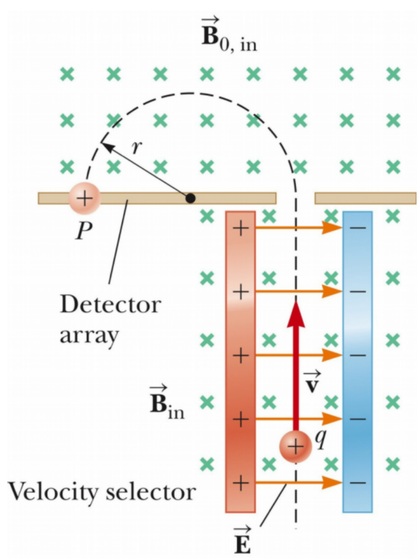
$$\left|\vec{F}_B\right| = \left|\vec{F}_E\right|$$

$$|q||\vec{v}|\left|\vec{B}\right| = |q|\left|\vec{E}\right|$$

$$\boxed{|\vec{v}| = \frac{\left|\vec{E}\right|}{\left|\vec{B}\right|}}$$

9.5 Mass Spectrometer

A mass spectrometer is used to identify different isotopes of an element, or more specifically the charge to mass ratio of the particle passed through it. The inside of a mass spectrometer looks like:

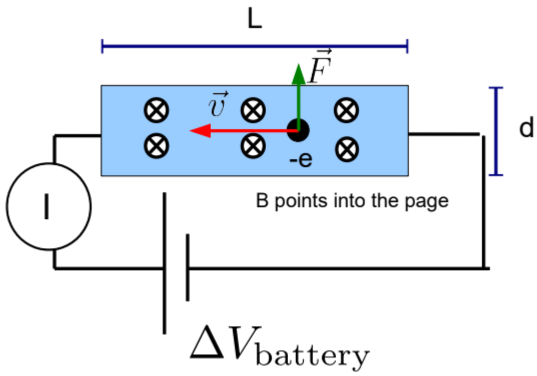


This device combines the concept of a cyclotron radius and a velocity selector. The particle is passed through a velocity selector, and then sent into a uniform magnetic field. The radius of the particles motion can be measured and so the following ratio can be evaluated:

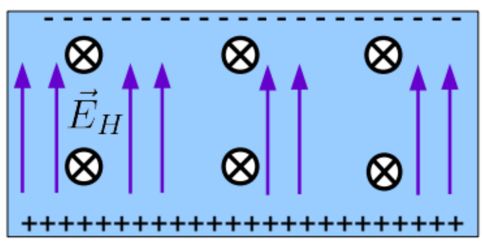
$$r = \frac{m|\vec{v}|}{|q|\left|\vec{B}\right|} \implies \frac{|q|}{m} = \frac{|\vec{v}|}{r\left|\vec{B}\right|}$$

9.6 The Hall Effect

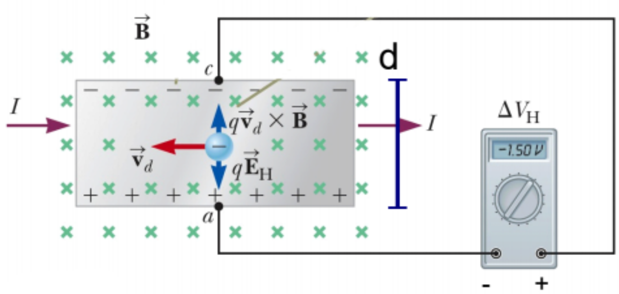
The Hall Effect occurs when a current is passed through a conductor, with a magnetic field that runs perpendicular to the current.



The charges will be deflected due to the magnetic force, which will cause a buildup of charge on one side of the conductor. This buildup of charge creates a new induced field, call the *Hall Electric Field*.



The charges are no longer deflected when the edges become sufficiently charged, such that there is a balance between the electric force and the magnetic force.



The equilibrium voltage across the conductor is called the *Hall Voltage* (ΔV_H). The induced field inside the conductor is the strength that balanced the magnetic and electric forces, defined by:

$$\left|\vec{E}_H\right| = |\vec{v}_d|\left|\vec{B}\right|$$

... and the voltage is expressed by:

$$|\Delta V_H| = |\vec{v}_d|\left|\vec{B}\right| d$$

... where d is the thickness of the conductor perpendicular to the current. If you can measure d, $\vec{B}$ and the Hall Voltage, you can experimentally determine the drift velocity. Note that when discussing conductors, L is the length along which current is flowing, t is the length along which the magnetic field is applied, and d is perpendicular to L and t. The *Hall Voltage* is related to the current through the conductor by:

$$|\Delta V_H| = \frac{I\left|\vec{B}\right| d}{nA|q|} = \frac{I\left|\vec{B}\right|}{nt|q|}$$

The *Hall Voltage* is useful for determining things like unknown magnetic fields, density of charges

9.7 Force on a Current Carrying Wire

We know that force on a single charge from a magnetic field can be described by:

$$\vec{F} = q(\vec{v}_d \times \vec{B})$$

... now we can multiply this by the total number of charges (N) which are in the wire, written as:

$$N = nAL$$

So in general, the total force on a wire of length L and cross sectional area A is:

$$\vec{F}_B = (qnA\vec{v}_d L) \times \vec{B}$$

$$\vec{F}_B = I\vec{L} \times \vec{B}$$

... where $\vec{L}$ is a vector with magnitude that is the length of the wire, and direction of conventional current. For wires there is no need to flip your hands for negative charges, as you point your fingers in the direction of the *current* and then curl your fingers in the direction of the magnetic field. If the wire is made up of multiple segments, or the magnetic field changes in certain parts of the wire, then add the force vector on each segment.

The magnetic field acting on two current carrying wires is described by:

$$\left|\vec{B}\right| = \frac{\mu_0 I_2}{2\pi a}$$

... where a is the distance between the wires and $\mu_0 = 4\pi \times 10^{-7}\frac{T\cdot m}{A}$. The magnitude of the force caused by the interaction of the two wires is:

$$\boxed{\left|\vec{F}\right| = \frac{\mu_0 I_1 I_2 \ell}{2\pi a}}$$

Questions sometimes as for *force per length*, and in that case you just divide the formula by length of wire. This force is attractive if the currents are in the same direction, and repulsive if they are in opposite directions.

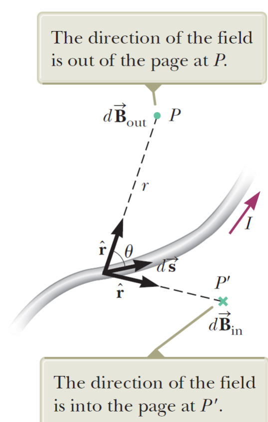
10 Chapter 29: Sources of the Magnetic Field

10.1 The Biot-Savart Law

The **Biot-Savart Law** describes the magnetic field induced ($d\vec{B}$) from a small segment of wire ($d\vec{s}$). It is a **law** that is always true. The law described the differential $d\vec{B}$ which can then be turned into $\vec{B}$ by:

$$\vec{B} = \int d\vec{B}$$

The law is written with the following image in mind:



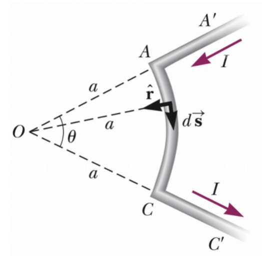
... and is written as:

$$\boxed{d\vec{B} = \frac{\mu_0 I}{4\pi} \frac{d\vec{s} \times \hat{r}}{|\vec{r}|^2}}$$

$\vec{r}$ is the vector which goes from the small segment of wire to the point of interest, while $d\vec{s}$ is the actual length of wire segment. I is the current through the wire segment, and μ_0 is the permeability of free space defined as:

$$\mu_0 = 4\pi \times 10^{-7} \frac{T \cdot m}{A}$$

The *Biot-Savart Law* has three important results which we will discuss now: Given the following situation :



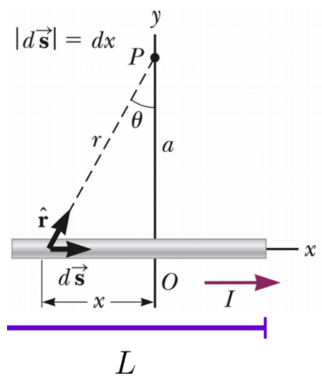
The magnitude of the magnetic field at point O is:

$$\boxed{\left|\vec{B}\right| = \frac{\mu_0 I}{4\pi a}\theta}$$

... or when $\theta = 2\pi$:

$$\left|\vec{B}\right| = \frac{\mu_0 I}{2a}$$

Given the following situation (distance a from the center of a wire of length L):



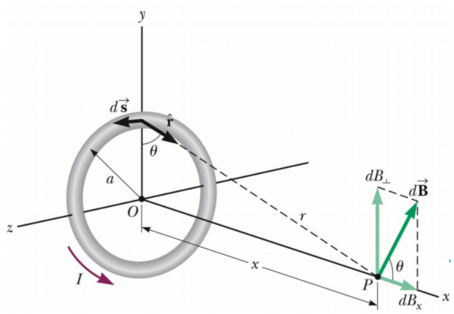
The magnitude of the magnetic field at point P is:

$$\boxed{\left|\vec{B}\right| = \frac{\mu_0 I L}{2\pi a\sqrt{L^2 + 4a^2}}}$$

... or when $L >> a$:

$$\left|\vec{B}\right| = \frac{\mu_0 I}{2\pi a}$$

Given the following situation (distance x from the center of a ring of current with radius a):



The magnitude of the magnetic field at point P is:

$$\boxed{\left|\vec{B}\right| = \frac{\mu_0 I a^2}{2(a^2 + x^2)^{3/2}}}$$

Conclusion

This concludes the course content from PCS 125. I hope that these notes were helpful, and that the final exam goes well for anyone reading! :)
- Adam Szava