Time Value of Money Part 1

BUS 312: Introduction to Finance

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Time value of money Part 1

- Valuation of cash flows
- Interest rates and compounding

Valuation of Cash Flows

We will cover…

- Simple and compound interest
- For single or multiple cash flows, calculate:
  - Future value
  - Present value
  - Interest rates: r
  - Number of periods: n

Simple Interest

$$I = P \times r \times n$$

- Where: I is the interest paid, P is the principal, r is the interest rate and n is the number of time periods
- A $100 investment earned 7 percent simple interest per year for five years
  - The total interest paid would be:

$$I = \$100 \times 7\% \times 5 = \$35$$

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Compound Interest

- Simple interest almost never applies. Instead, interest is typically earned on interest
- The future value is the amount to which an investment will grow after earning interest

$$Future\ Value = Investment \times (1 + r)^n$$

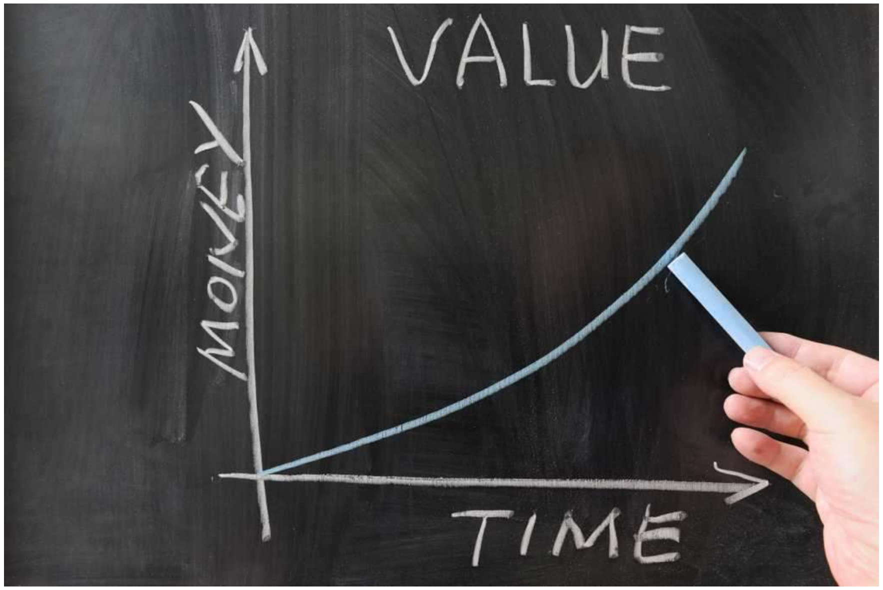
- The expression $(1 + r)^n$ refers to compound interest, which is interest earned on interest at the rate r for n periods

Compound Interest

- An investment of $100 for five years at 7 percent interest, compounded annually would be:

$$I = \$100 \times (1 + 7\%)^5 = \$140.26$$

Time Value of Money



- But, how much more is it worth?

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More formally… Future Value of Money

- Future Value: the amount an investment is expected to be worth after a given number of periods

$$FV = PV \times (1 + r)^n$$

- Where:
  - PV is the value of the investment today
  - r is the effective interest rate per period
  - n is the number of periods

Future Value Example

- You invest $1,000 at 4% interest per year and don't touch the money for 5 years. At the end of that time, how much is the investment worth?

$$FV = PV \times (1 + r)^n$$
$$FV = \$1{,}000 \times (1 + 4\%)^5$$
$$FV = \$1{,}216.65$$

Present Value and Discounting

- It's often even more useful to go in the other direction, taking some future amount and determining its value today
- This is called discounting, where we find the present value

$$PV = FV \times (1 + r)^{-n}$$

- Where:
  - FV is the value of the future cash flow

Multiple periods PV example

- Your parents promise to buy you a $30,000 car when you graduate three years from now. The bank is willing to pay 2.5% interest per year. How much should they save today?

$$PV = FV \times (1 + r)^{-n}$$
$$PV = \$30{,}000 \times (1 + 2.5\%)^{-3}$$
$$PV = \$27{,}857.98$$

Manipulating the formula

- To isolate r, the interest rate per period:

$$FV = PV \times (1 + r)^n$$

$$\frac{FV}{PV} = (1 + r)^n$$

$$r = \left(\frac{FV}{PV}\right)^{\frac{1}{n}} - 1$$

Manipulating the formula

- To isolate n:

$$FV = PV \times (1 + r)^n$$

$$\frac{FV}{PV} = (1 + r)^n$$

$$\ln\left(\frac{FV}{PV}\right) = n \ln(1 + r)$$

$$n = \frac{\ln\left(\frac{FV}{PV}\right)}{\ln(1 + r)}$$

Multiple Cash Flows

- Our formulas for present value and future value up to this point have assumed one cash flow
- What if we have to value several cash flows?

Dealing with Multiple Cash Flows

1. Draw a timeline
2. Determine when each cash flow occurs
3. Determine whether you want a present value or a future value
4. Either discount each cash flow back, or compound each cash flow forward

Interest Rates and Compounding

We will cover…

- Effective Annual interest rates (EAR)
- Annual Percentage interest rates (APR)
- Effective periodic interest rates
- Effect of compounding frequency

Effective Annual Interest Rates

- Interest rates may be calculated for any time interval
  - E.g. day, month, quarter, year.
- The effective annual interest rate (EAR) is the interest rate that would actually apply if interest were calculated once per year:

$$(1 + EAR) = (1 + r)^m$$

- Where r is the rate for the period and m is the number of periods of compounding **in one year**

Effective Annual Interest Rates

- What is the EAR for a 3-month interest rate of 5%?
  - $r = 5\%$
  - Period: 3-months
  - $m = 4$ number of periods (3-months) of compounding in one year

$$EAR = (1 + r)^m - 1 = (1 + 5\%)^4 - 1 = 22\%$$

APR Interest Rates

- Canada's Bank Act requires that interest rates for short time periods be quoted as annual rates by multiplying the per-period rate by the number of periods in a year.

$$APR = m \times r$$

- The resulting annual percentage rates (APRs) do not recognize the effect of compound interest.

APR and EAR

What is the EAR of a 12% APR rate, compounded monthly?

$$APR = m \times r = 12\% = 12 \times r \Rightarrow r = 1\%$$
$$EAR = (1 + 1\%)^{12} - 1 = 12.68\%$$

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APR and EAR

- To compute EAR from APR:

$$EAR = \left(1 + \frac{APR}{m}\right)^{m} - 1$$

- Where
  - $APR = m \times r$
    - r: period interest rate
    - m: # of compounding periods per year

Periodic Effective Rates

- Suppose you plan to repay a loan in one month, not one year
  - What is the effective monthly rate of interest?
- We now need a formula which will enable us to convert APR rates into periodic effective rates
  - That is, effective interest rates for a variety of time horizons, not just for one year

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Periodic Effective Rates

- To compute EAR from APR:

$$r_{Periodic} = \left(1 + \frac{APR}{m}\right)^{\frac{m}{k}} - 1$$

- Where
  - $r_{Periodic}$: effective period interest rate
  - $APR = m \times r$
    - r: period interest rate
    - m: # of compounding periods per year
  - k: # of payment periods per year

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Periodic Effective Rates

What is the monthly Periodic Effective Rates of a 40% APR rate, compounded daily?

$$r_{Periodic} = \left(1 + \frac{APR}{m}\right)^{\frac{m}{k}} - 1$$

$$r_{Periodic} = \left(1 + \frac{40\%}{365}\right)^{\frac{365}{12}} - 1 = 3.39\%$$

Interest Rate Quoting Conventions

- APR rates are frequently quoted as $[APR, m]$, which implies m compounding periods per year
- Example:
  - The rate [10%, 2] means that 5% interest is compounded every 6-month period
  - So, the EAR is 10.25%
    - $EAR = \left(1 + \frac{APR}{m}\right)^m - 1 = \left(1 + \frac{10\%}{2}\right)^2 - 1 = 10.25\%$

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How do I recognize an APR rate?

- It is quoted or you are told it is an APR
- It is annual (a monthly rate cannot be an APR)
- It is in a contract or advertising
- It is given with the compounding frequency

Changing the Compounding Frequency

- Let's examine the effect of changing the compounding frequency
  - As interest is calculated at shorter and shorter intervals, interest is earned on interest more frequently
  - As a result, the effective annual rate of interest increases
- The following table presents these calculations
  - We will assume that $APR = 12\%$ and $k = 1$ (EAR)
  - Then, we can observe the effect of changing the # of compounding periods per year m

Assume that APR = 12% and k = 1 (EAR) Observe the effect of increasing m …

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Compounding Frequency	Periods per Year	Per Period Interest Rate	FV of $1 invested for 1 year	EAR
Yearly	1	12%	$(1+12\%)^{1}$	12%
Semiannually	2	6%	$(1+0.6\%)^{2}$	12.36%
Quarterly	4			
Monthly	12	1%	$(1+1\%)^{12}$	12.68%
Weekly	52			
Daily	365	0.03%	$(1+0.03\%)^{365}$	12.747%
Continuously	∞		$e^{12\%}$	12.750%

Continuous Compounding

- As m increases, the EAR increases, but does it increase without bound?
- We can determine this bound by taking the ultimate step:
  - Assume that interest is compounded continuously
    - Thus $m \to \infty$!

Continuous Compounding

- For continuous compounding, the effective rate is calculated as:

$$r_{periodic} = e^{APR \times t} - 1$$

- Where:
  - e = 2.718 …
    - The base for natural logarithms
  - t: # of years in the period you wish to calculate
    - 6 months implies $\mathrm{t} = 0.5$
    - 3 years implies $\mathrm{t} = 3$

Converting between effective rates

$$\left(1 + r_{quarterly}\right)^4 = (1 + r_{annual})$$

$$\left(1 + r_{quarterly}\right)^2 = (1 + r_{semi-annual})$$

$$\left(1 + r_{monthly}\right)^3 = (1 + r_{quarterly})$$

$$\left(1 + r_{daily}\right)^{365} = (1 + r_{annual})$$

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Conclusion

- Define future value and present value.
- Discount future cash flows to determine their present value.
- Distinguish between effective annual interest rates (EAR) and nominal APR rates.
- Define and calculate effective periodic rates of interest.
- Determine the effect of changing the frequency of compounding.