DISCRETE MATHEMATICS FOR COMPUTING

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MAT1348C Assignment 3

due: Thursday, April 8 by 11:30 pm

INSTRUCTIONS

- You must upload your assignment on Brightpsace on or before **Thursday, January 28 by 11:30 pm**. **Late assignments will not be accepted.** Make sure you submit your whole assignment without missing any part.
- Please print this document, including the cover page, and write your answers in the space provided. You may write on the backs of pages or insert extra pages if necessary, so long as the assignment is clearly organized, with solutions in the same order as the questions, and **scan whole including the cover page. Be sure without missing any part.**
- You must hand in a legible, organized and properly stapled assignment. If it is too difficult to read your solutions, then you may get zero.
- Solutions must include all relevant steps and justifications where appropriate. If you only write the final answer without explanation, then you may not receive full marks.
- The maximum points possible = 20 points.
- **Please just upload a PDF file including your whole work, otherwise your assignment may not be marked completely.**

1.

FAMILY NAME:	STUDENT NUMBER:
FIRST NAME: Solutions	†SIGNATURE:

Table of Points for marking purposes. *Do not write in this table.*

	Q1	Q2	Q3	Q4	BONUS	TOTAL
max points possible	4 pts	6 pts	4 pts	6 pts	+3 pts	20 points
points obtained						

Q1. [4 points] Suppose we have a specially -adapted typewriter for Monkeys to use that has only 15 keys:

$$A \ B \ C \ D \ E \ K \ L \ M \ N \ O \ P \ W \ X \ Y \ Z$$

Important All of your solutions must be clearly justified with explanations and you must write each of your final answer in two ways.

- As a formula involving only products, sums, differences, or powers.
- As one final concrete number (where you might need to make use of a calculator)

Using the monkey typewriter, how many 13-letter words are there consisting of 13 **distinct** letters that begin with 'ABC' **or** end with 'MONKEY'? *(this is an inclusive or) (letters may not be repeated.)*

Let $S = \{$13 - Letter words (distinct) that begin with 'ABC'$\}$

A B C _ _ _ _ _ _ _ _ _ _

1 1 1 12 11 10 9 8 7 6 5 4 3 $\quad |S| = P(12,10)$

we cannot repeat letters and we already used A, B, C.

Let $T = \{$13 - letter words (13 distinct letters) that end with 'MONKEY'$\}$

_ _ _ _ _ _ _ M O K E Y

9 8 7 6 5 4 3 1 1 1 1 1 $\Rightarrow |T| = P(9,7)$

then $S \cap T = \{$13-Letter words (distinct) that begin with 'ABC' and end with 'MONKEY'$\}$

A B C _ _ _ _ M O N K E Y

1 1 1 6 5 4 3 1 1 1 1 1 1 $\Rightarrow |S \cap T| = P(6,4)$

then $S \cup T = \{$13 - Letter words (13 distinct letters) that begin with 'ABC' or end with 'Monkey'$\}$

BY P.I.E.

$|S \cup T| = |S| + |T| - |S \cap T| = P(12,10) + P(9,7) - P(6,4)$

$= 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 + 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 - 6 \cdot 5 \cdot 4 \cdot 3$

$= 239681880$

$\therefore$ there are 239681880 such 13-letter words.

Q2. [6 points] **Induction.** For each integer $n \geq 1$, define the sum $S(n)$ as follows:

$$S(n) = \sum_{n=1}^{n} \frac{1}{(2i-1)(2i+1)} = \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots + \frac{1}{(2n-1)(2n+1)}. \quad .$$

For example $S(1) = \sum_{n=1}^{1} \frac{1}{(2i-1)(2i+1)} = \frac{1}{1\cdot 3} = \frac{1}{3}$ and $S(2) = \sum_{i=1}^{2} \frac{1}{(2i-1)(2i+1)} = \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} = \frac{2}{5}$

– **i.** Compute each of the following values of $S(n)$ and write your final answer as a fraction in lowest terms.

$S(3) = \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} = \frac{3}{7}$

$S(4) = \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \frac{1}{7\cdot 9} = \frac{4}{9}$

$S(5) = \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \frac{1}{7\cdot 9} + \frac{1}{9\cdot 11} = \frac{5}{11}$

– **ii.** Do you see a general pattern for the simplified value of $S(n)$? Write your guess to complete the proposition $P(n)$ below.

For each integer $n \geq 1$, let $P(n)$ be the proposition defined as follows:

$$P(n): \quad S(n) = \sum_{i=1}^{n} \frac{1}{(2i-1)(2i+1)} = \frac{n}{2n+1}$$

– **iii.** Prove that the proposition $P(n)$ is true for all $n \geq 1$ using a **Proof by induction.**

*You must clearly state your **Induction Hypothesis** and indicate when it is used during the proof of your **Induction Step.** As usual you must declare what each variable in your solution represents and make it clear whether each step of your proof is an assumption, something you are about to prove, or something that follows from a previous step or definition, etc. Attach additional paper if necessary*

BI $n_0 = 1$: LS of $P(1) = \sum_{i=1}^{1} \frac{1}{(2i-1)(2i+1)} = \frac{1}{1\cdot 3} = \frac{1}{3}$

RS of $P(1) = \frac{1}{2(1)+1} = \frac{1}{3}$ Since LS = RS, $P(1)$ is True.

IS. Let k be an Integer such that $k \geq 1$ (base Value)

I.H. Assume $P(k)$ is True. that is assume $\sum_{i=1}^{k} \frac{1}{(2i-1)(2i+1)} = \frac{k}{2k+1}$

(goal: prove $P(k+1)$ follows from $P(k)$)

$P(k+1)$ says $\displaystyle\sum_{i=1}^{k+1} \frac{1}{(2i-1)(2i+1)} = \frac{k+1}{2(k+1)+1}$ ← this is what we want to prove.

$$\text{L.S of } P(k+1) = S(k+1)$$

$$= \sum_{i=1}^{k+1} \frac{1}{(2i-1)(2i+1)}$$

$$= \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \cdots + \frac{1}{(2k-1)(2k+1)} + \frac{1}{(2(k+1)-1)(2(k+1)+1)}$$

$$= S(k) + \frac{1}{(2k+1)(2k+3)}$$

By the I.H. $$= \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)}$$

$$= \frac{k}{2k+1} + \frac{1}{2}\left[\frac{1}{2k+1} - \frac{1}{2k+3}\right]$$

$$= \frac{1}{2} - \frac{1}{2}\,\frac{1}{2(k+1)+1}$$

$$= \frac{2(k+1)}{2\left[2(k+1)+1\right]}$$

$$= \frac{k+1}{2(k+1)+1} = \text{R.S of } P(k+1).$$

∴ We proved $P(k) \longrightarrow P(k+1)$. Since $P(1)$ is True it follows from the principle of Mathematical Induction that $P(n)$ is True for all $n \geq 1$.

thus $\displaystyle\sum_{i=1}^{n} \frac{1}{(2i-1)(2i+1)} = \frac{n}{2n+1}$ for all Integers $n \geq 1$.

Q3. [4 points] a. Complete the following statement of the Handshaking Theorem:

The Handshaking Theorem:
If $G = \big(V(G), E(G)\big)$ is any graph, then $\cdots$.

$$\sum_{u \in V(G)} \deg_G(u) = 2|E(G)|$$

- b. Let G be a graph with 12 vertices and 25 edges.

Suppose each vertex of G has degree 0, 3, or 7.
If G has exactly 5 vertices of degree 7, how many isolated vertices does G have?

Number of isolated vertices in G **:** 2.

Let : $a = \begin{pmatrix} \text{\# Vertices} \\ \text{of degree } 0 \end{pmatrix}$ $b = \begin{pmatrix} \text{\# Vertices} \\ \text{of degree } 3 \end{pmatrix}$ $c = \begin{pmatrix} \text{\# Vertices} \\ \text{of degree } 7 \end{pmatrix}$

then

$a+b+c = 12$ and $a(0) + b(3) + c(7) = \sum_{u \in V(G)} \deg_G(u) = 2|E(G)| = 50$

$\therefore$ $a+b+c=12$ and $3b + 7c = 50$. We are given that

$c = 5 \Rightarrow 3b = 15$

$b = 5$

$\therefore a = 12 - 5 - 5 = 2$.

$\therefore$ G has 2 isolated vertices.

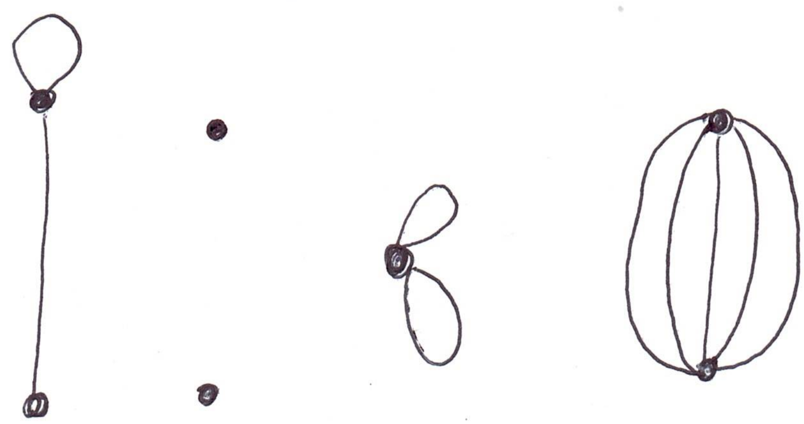
Q4. [6 points] Consider the following sequences:

$$(0, 0, 1, 3, 4, 5, 5)$$

Does there exist a graph with the above degree sequences? *Circle:* YES NO

If you circled **Yes**, draw a picture of such a graph; otherwise , explain why no such graph exists.

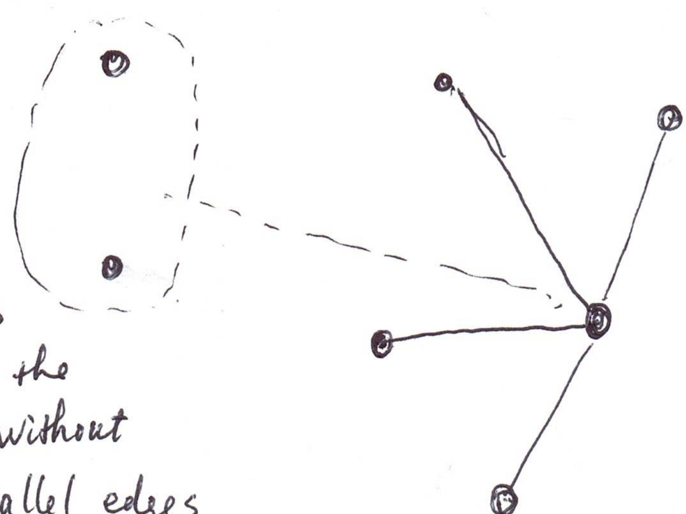
(# Vertices) = 7. (# edges) $= \frac{1}{2}(0+0+1+3+4+5+5) = 9$



Does there exist a **simple** graph with the above degree sequences *Circle:* YES NO

If you circled **Yes**, draw a picture of such a graph; otherwise, explain why no such graph exists.

With these 2 vertices of degree 0. It is impossible to have any vertices of degree 5 among the other 5 vertices without using loops or parallel edges.



(BONUS) [+3 bonus points] Find the exact numbers to answer each of the following questions. You do not need to show any justification for this, but you will only earn the bonus points if at least 4 out of 5 of your answers are correct.

i. How many 5-letter passwords(letter may be repeated) can be created from the monkey type writer from Q1 if it is forbidden for a password to begin and end with the same letter?

Answer: 708750

ii. Using the monkey typewriter from Q1, how many 8-letter words are there consisting of 8 **distinct** letters that contain the word 'APE' (as 3 consecutive)letters within the 8-letter word?

Answer: 570240

iii. In a strange experiment involving N monkeys, we let each monkey create one 2 -letter password with the monkey typewriter(letters may be repeated). What is the exact minimum number N of monkeys needed in order to **guarantee** that we will see the same 2 letter password created by at least 5 of the N monkeys? (each monkey is free to choose whatever random password it wishes, regardless of whatever passwords the other monkeys have chosen).

The minimum number of the monkeys needed is N =: 901

iv. Suppose that a group of 9 inhabitants of the Island of Knights & Knaves need to create a committee of 6 members so that the number of knights on the committee is **greater than or equal to** the number of knaves, In how many ways is it possible to create such a committee given the following information:

-of these 9 inhabitants, 5 are knights, namely A, B, C, D, and E, and
-of these 9 inhabitants, the other 4 are knaves, namely W, X, Y, and Z.

Answer: 74

v. What is the coefficient of $a^{-1}b^{13}$ in the expansion of $(2a^3 - \frac{b}{a})^{17}$?

Answer: -38080

i). The total # of 5-letter passwords that can be created is 15^5

How many forbidden passwords are there?

$$\underset{15}{\_}\ \underset{15}{\_}\ \underset{15}{\_}\ \underset{15}{\_}\ \underset{1}{\_}$$

choices for start letter → 15 15 15 15 1 ← way to repeat the first letter

$\Rightarrow 15^4$ forbidden passwords.

$\Rightarrow$ total # of permitted passwords $= 15^5 - 15^4 = 708750$

ii) choose location of 'APE' in one of 6 ways

APE _ _ _ _ _

_ APE _ _ _ _

_ _ APE _ _ _

_ _ _ APE _ _

_ _ _ _ APE _

_ _ _ _ _ APE

choose 5 other the remaining $15-3=12$ letters to arrang in the remaining 5 spaces in one of $P(12,5)$ ways.

$\Rightarrow$ there are $(6)\cdot P(12,5) = 6\cdot 12\cdot 11\cdot 10\cdot 9\cdot 8 = 570240$ such 8-letter words.

iii). there are $k = 15^2 = 225$ possible different 2-letter passwords

In the worst case, if we had $N = 225\cdot 4 = 900$ monkeys, it is possible that each one of the 225 possible passwords is created by 4 of the monkeys, so we never see the same password at least 5 times.

with $N = 225\times 4 + 1 = 901$ monkeys we are guaranteed to see at least one of the passwords created by at least 5 of the monkeys.

This is guaranteed by the generalized pigeonhole principle with $k = 225$ boxes $N = 901$ objects since GPP guarantees at least one box will contain at least $\lceil \frac{N}{k} \rceil = \lceil \frac{901}{225} \rceil = 5$ objects

IV). To have 6 members with # knights $\geq$ # knaves, we have 3 possibilities.

* create a committee with 3 knights + 3 knaves in one of $\binom{5}{3}\binom{4}{3}$ ways.
* create a committee with 4 knights + 2 knaves in one of $\binom{5}{4}\binom{4}{2}$ ways
* create a committee with 5 knights + 1 knaves in one of $\binom{5}{5}\binom{4}{1}$ ways.

$\Rightarrow$ In total there are $\binom{5}{3}\binom{4}{3}+\binom{5}{4}\binom{4}{2}+\binom{5}{5}\binom{4}{1}$

$= 10\cdot 4 + 5\cdot 6 + 1\cdot 4$

$= 74$ such committees.

$$\left(2a^3-\frac{b}{a}\right)^{17} = \sum_{j=0}^{17}\binom{17}{j}(2a^3)^j\left(-\frac{b}{a}\right)^{17-j}$$
$$= \sum_{j=0}^{17}\binom{17}{j}2^j a^{3j}\cdot(-1)^{17-j}\cdot b^{17-j}\cdot a^{j-17}$$
$$= \sum_{j=0}^{17}\binom{17}{j}2^j(-1)^{17-j}a^{4j-17}b^{17-j}$$

For the coefficient of $a^{-1}b^{13}$ we need the index $j\in\{0,1,2,\cdots,17\}$ such that $4j-17=-1$ and $17-j=13$. $\therefore j=4$.

$\Rightarrow$ the coefficient of $a^{-1}b^{13}$ is $\binom{17}{4}2^4(-1)^{13} = -\binom{17}{4}\cdot 2^4$

$= \frac{17\cdot16\cdot15\cdot14}{4\cdot3\cdot2\cdot1}\times 2^4\cdot(-1) = -38080$