UIT2206: The Importance of Being Formal

# Assignment Week 04: Propositional Logic I

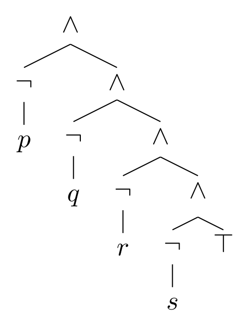
1. (5 marks) Draw the parse tree for the following formula:

$$((\neg p) \wedge ((\neg q) \wedge ((\neg r) \wedge ((\neg s) \wedge \top))))$$

List all sub-formulas of the expression.

**Solution**

The parse tree looks as follows:




There are 13 subformulas:

- $((\neg p) \wedge ((\neg q) \wedge ((\neg r) \wedge ((\neg s) \wedge \top))))$
- $(\neg p)$
- $p$
- $((\neg q) \wedge ((\neg r) \wedge ((\neg s) \wedge \top)))$
- $(\neg q)$
- $q$
- $((\neg r) \wedge ((\neg s) \wedge \top))$
- $(\neg r)$
- $r$

- $((\neg s) \land \top)$
- $(\neg s)$
- $s$
- $\top$

2. (3 marks) According to the operator precedences in Convention 1 (page 7), the following formula has a unique reading.

$$\neg p \land q \to \neg r \lor \neg p \to r$$

Indicate this reading by writing all parentheses, according to Definition 1.

**Solution**

$$(((\neg p) \land q) \to (((\neg r) \lor (\neg p)) \to r))$$

3. (3 marks) Consider the special case of propositional logic where the set of atoms $A$ is empty, and where there are no binary operations. Is the set of resulting formulas still infinite? Describe the set of resulting formulas using English or a mathematical notation such that the reader understands what formulas it contains.

**Solution**

The formulas in this special case are either $\top$, or $\bot$, or arbitrarily long sequences of $\neg$ followed by $\top$ or $\bot$. One could write this set (by appealing to the readers intelligence to spot the pattern) as follows:

$$\{\top, \bot, \neg\top, \neg\bot, \neg\neg\top, \neg\neg\bot, \ldots\}$$

4. (3 marks) Give the result of evaluating the following formula with respect to the valuation $v : v(p) = F, v(q) = T, v(r) = T, v(s) = F$. You need to submit only the result. You can apply the method involving the syntax tree shown in class, or the method explained in the notes on page 9 (Example 4).

- $(\neg q \lor r) \land (r \to \neg q)$
- $p \to q \to s$
- $(\bot \to \bot) \land (\top \to \top)$

**Solution**

F, T, T

5. (3 marks) Is the following formula valid?

$$(p \land \neg q) \to (q \lor \neg p)$$

Show how you arrived at your answer.

**Solution**

Validity would require that the last column of the following truth table only contains T:

| $p$ | $q$ | $p \wedge \neg q$ | $q \vee \neg p$ | $(p \wedge \neg q) \rightarrow (q \vee \neg p)$ |
|---|---|---|---|---|
| F | F | F | T | T |
| F | T | F | T | T |
| T | F | T | F | F |
| T | T | F | T | T |

Note that the last column does not only contain T; there is one F. Therefore, the formula is not valid.

6. (4 marks) Consider formulas of the following form:

$$p_1 \rightarrow p_2 \rightarrow p_3 \rightarrow p_4 \rightarrow \cdots p_n \rightarrow \top$$

Are all such formulas valid? Show how you arrived at your answer.

**Solution**

Let us call the described set $P$. Let us define a set $P'$ as follows:

> $P'$ is the smallest set that satisfies the following two rules:
> - $\top$ is in the set.
> - If a formula $\phi$ is in the set, then $p \rightarrow \phi$ is also in the set.

It is easy to see that $P \subseteq P'$. So if we show that all elements of $P'$ are valid, we are done. Let $V$ be the set of all valid formulas. We will show that $P' \subseteq V$, which means that all elements of $P'$ are valid.

Our strategy for showing $P' \subseteq V$ is that we show that $V$ satisfies both given rules. Since $P'$ is the smallest set that satisfies both rules, we must have $P' \subseteq V$.

So what remains is to show that $V$ satisfies both rules:

- $\top$ is in the set: We need to show that $\top$ is valid. This is the case by the definition of the semantics of $\top$.
- If a formula $\phi$ is in the set, then $p \rightarrow \phi$ is also in the set: Assume a valid formula $\phi$. It will evaluate to $T$ for every valuation. Consider $p \rightarrow \phi$. According to the semantics of $\rightarrow$, this formula can evaluate to $F$ only if $p$ evaluates to $T$ and $\phi$ evaluates to $F$. However, $\phi$ is valid and therefore can never evaluate to $F$. Therefore, $p \rightarrow \phi$ evaluates to $T$ for every valuation, and thus $p \rightarrow \phi$ is valid, and thus included in the set $V$.

A proof of this kind that exploits the fact that set is defined inductively, as the smallest set that fulfills a given number of rules, is called a *proof by induction.*