1. Find all vectors in $\mathbb{R}^3$ which are perpendicular to both $(-1,1,5)$ and $(2,1,2)$. (1)

`cross (X) the correct answer:`

- A $(2,-4,2)$
- B $(t+1,-4,t+1) \mid t \in \mathbb{R}$
- C $(-t,0,2t) \mid t \in \mathbb{R}$
- D $(-t,4t,-t) \mid t \in \mathbb{R}$
- E $(1,1,1)$
- F $(4,-12,4)$

Such vectors will be parallel to

$$u \times v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & 5 \\ 2 & 1 & 2 \end{vmatrix} = (-3,12,-3).$$

Hence $\boxed{\text{D}}$ is correct

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2. If $u = (3,3,6)$ and $v = (2,-1,1)$ then the length of the projection of u along v is: (1)

`cross (X) the correct answer:`

- A $\frac{3\sqrt{2}}{2}$
- B $\frac{3\sqrt{6}}{2}$
- C $\frac{2\sqrt{6}}{3}$
- D $\frac{\sqrt{6}}{2}$
- E 0
- F $\frac{2\sqrt{2}}{3}$

Solution: Since $||proj_v u|| = \frac{|u \cdot v|}{||v||^2}||v|| = \frac{|u \cdot v|}{||v||} = \frac{9}{\sqrt{6}} = \frac{3\sqrt{6}}{2}$. So the correct answer is $\boxed{\text{B}}$.

3. Mark whether each of the following statements is TRUE or FALSE in the respective box. (each correct answer is 1/4pt)

- A homogeneous system of linear equations can have infinitely many solutions.

  ANSWER: [ ] TRUE

- It is possible that a system of linear equations with coefficients in $\mathbb{R}$ has exactly 2 solutions.

  ANSWER: [ ] FALSE

- If a linear system is inconsistent, then it can not be homogeneous.

  ANSWER: [ ] TRUE

- There exists a linear system of three equations such that its coefficient matrix has rank 5.

  ANSWER: [ ] FALSE

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4. If the coefficient matrix A in a homogeneous system in 21 variables of 17 equations is known to have rank 11, how many parameters are there in the general solution? (1)

`cross (X) the correct answer:`

[A] 11

[B] 10

[C] 6

[D] 21

[E] 17

[F] 4

Solution: Homogeneous system is consistent. So the number of parameters is 21 - 11=10. So the correct answer is [B].

5. Suppose $e, f \in \mathbb{R}$ and consider the linear system in x, y and z:

$$\begin{cases} 2x - 2y + ez & = f \\ x + z & = -1 \\ 3x + y + 2z & = -1 \end{cases}$$

5(a) If $(A \mid b)$ is the augmented matrix of the system above, find the rank of A and the rank of $(A \mid b)$ for **all** values of e and f. (2)

(justify your answers)

Solution: We have

$$(A \mid b) = \begin{pmatrix} 2 & -2 & e & | & f \\ 1 & 0 & 1 & | & -1 \\ 3 & 1 & 2 & | & -1 \end{pmatrix} \overset{Gauss\ elimination}{\longrightarrow} \begin{pmatrix} 1 & 0 & 1 & | & -1 \\ 0 & 1 & -1 & | & 2 \\ 0 & -2 & e-2 & | & f+2 \end{pmatrix}$$

$$\overset{Gauss\ elimination}{\longrightarrow} \begin{pmatrix} 1 & 0 & 1 & | & -1 \\ 0 & 1 & -1 & | & 2 \\ 0 & 0 & e-4 & | & f+6 \end{pmatrix}$$

Hence,

$$rank\ A = \begin{cases} 2 & if\ e = 4 \\ 3 & if\ e \neq 4 \end{cases}$$

$$rank\ (A \mid b) = \begin{cases} 2 & if\ e = 4\ and\ f = -6 \\ 3 & otherwise \end{cases}$$

5(b) Using part (a), find all values of e and f so that this system has

(i) a unique solution (1)

Solution: it has a unique solution $\iff rank\, A = rank\,(A \mid b) = 3 \iff e \neq 4$.

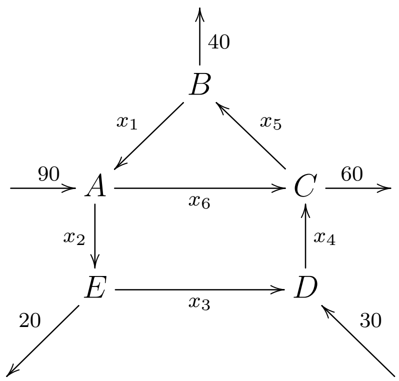
(ii) infinitely many solutions (1)

Solution: it has infinitely many solutions $\iff rank\, A = rank\,(A \mid b) < 3 \iff e = 4$ and $f = -6$.

(iii) no solutions (1)

Solution: it has no solutions $\iff rank\, A < rank\,(A \mid b) \iff e = 4$ and $f \neq -6$.

6. Consider the network of streets with intersections A, B, C, D and E below. The arrows indicate the direction of traffic flow along the one-way streets, and the numbers refer to the exact number of cars observed to enter or leave A, B, C, D and E during one minute. Each x_i denotes the unknown number of cars which passes along the indicated streets during the same period.



6(a) Write down a system of linear equations which describes the traffic flow **together with all the constrains on the variables** x_i, $i = 1, \ldots, 6$. (1)

(Do not perform any operations on your equations: this is done for you in (b). Do not simply copy out the equations implicit in (b). You will not get any marks if you do this)

Solution:

Intersection	Flow in = Flow out
A	$90 + x_1 = x_2 + x_6$
B	$x_5 = x_1 + 40$
C	$x_4 + x_6 = x_5 + 60$
D	$30 + x_3 = x_4$
$E.$	$x_2 = x_3 + 20$

Oneway streets give constrains $x_i \geq 0$, $i = 1, \ldots, 6$. Since each x_i is a number of cars, x_i has to be an integer.

6(b) The reduced row-echelon form of the augmented matrix of the system in part (a) is

$$\left(\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & -1 & 0 & -40 \\ 0 & 1 & 0 & 0 & -1 & 1 & 50 \\ 0 & 0 & 1 & 0 & -1 & 1 & 30 \\ 0 & 0 & 0 & 1 & -1 & 1 & 60 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Give the general solution of this system (1)

(Ignore the constrains from (a) at this point)

Solution: Set $x_5 = s$ and $x_6 = t$ to be parameters. Then

$$\begin{aligned} x_1 &= -40 + s \\ x_2 &= 50 + s - t \\ x_3 &= 30 + s - t \\ x_4 &= 60 + s - t \\ x_5 &= s \\ x_6 &= t \end{aligned}$$

6(c) If the road ED was closed in the middle due to roadwork, find the minimum flow along the road AC **using your results from (b)** (2)

(you must justify all your answers: correct answer without justification is 1pt only)

Solution: ED is closed $\iff x_3 = 0 \iff s - t = -30$. Then assuming the constrains $x_i \geq 0$ we obtain the system of inequalities

$$\begin{cases} x_1 &= -40 + s \geq 0 \iff s \geq 40 \\ x_2 &= 50 - 30 = 20 \\ x_3 &= 0 \\ x_4 &= 60 - 30 = 30 \\ x_5 &= s \geq 0 \\ x_6 &= s - 30 \geq 0 \iff s \geq 30 \end{cases}$$

The flow along AC is $x_6 = s + 30$ so that $x_6 \geq 70$. The minimum flow along x_6 is 70.

7. Mark whether each of the following statements is TRUE or FALSE in the respective box. (each correct answer is 1/4pt)

- There exist two non-zero matrices A and B such that $A \cdot B$ is the zero-matrix.

  ANSWER: [ ] TRUE

- For any two 3×3 matrices A and B, we have $(A+B)^2 = A^2 + 2AB + B^2$.

  ANSWER: [ ] FALSE

- Multiplying a 3×2-matrix A by a 2×3-matrix B one gets a 3×3-matrix AB.

  ANSWER: [ ] TRUE

- For any two matrices A and B we have $AB = BA$.

  ANSWER: [ ] FALSE

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8. If $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$, and B is a 3×5 matrix,

then the second row of the matrix $A \cdot B$ is (1)

`cross (X) the correct answer`:

[A] the same as the first row of B

[B] the sum of the first and the second rows of B

[C] the sum of the first, the second and the third rows of B

[D] the sum of the first and the third rows of B

[E] the same as the second row of B

[F] the sum of the second and the third rows of B

Solution: Write $B = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}$ in block form, where r_i is the i-th row of B. (1) Then

$$A \cdot B = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} = \begin{pmatrix} r_1 \\ r_1 + r_3 \\ r_1 + r_2 + r_3 \end{pmatrix}$$

So the correct answer is [D].

The last page (use it for computations)