Université d'Ottawa · University of Ottawa

Faculté des sciences
Mathématiques et de statistique

Faculty of Science
Mathematics and Statistics

MAT 2141, Fall 2017 – Midterm Exam **Instructor: A. Tikuisis**

Last name: ______________________

First name: ______________________

Signature: ______________________

Student number:

Read the following instructions:

- Not allowed: Calculators, textbooks, course notes, cell phones.
- Do not detach the pages of this examination.
- You may use the back of the pages as scrap paper for calculations, or to answer questions if you run out of space on the front side.
- You must give clear and complete solutions, with calculations, explanations and justifications. Make sure that your answer is clearly indicated.

THIS SPACE IS RESERVED FOR THE MARKER:

Question	1	2	3	4	5	6	Total
Mark							
Out of	5	5	4	4	4	3	25

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1. Let V be a vector space over a field F, other than the zero space. Label the following statements as true (T) or false (F). (5)

☐ If $v \in V$ is nonzero and $c \in F$ satisfies $cv = 0$ then $c = 0$.

☐ V must contain a linearly independent set.

☐ V must contain infinitely many subspaces.

☐ If $T : V \to V$ is a linear map and $W \subseteq V$ is a subspace then $T(W)$ is a subspace.

☐ If $S_1 \subset V$ is a basis of V and $S_2 \subseteq V$ is a set which spans V, then $S_1 \subseteq S_2$.

T
T
F
T
F

2. Label the following statements as true (T) or false (F). (5)

☐ In every field we have $1+1 \neq 0$.

☐ If F is a field then $M_2(F)$, with addition and matrix multiplication, is never a field.

☐ There exists a field F and some nonzero element $x \in F$ such that $x^3 = 0$.

☐ If F is a field and $V \subseteq F$ is a subset which is closed under addition and (field) multiplication, then V is a vector space over F.

☐ If F is a field, then there always exists a nonzero vector space V over F.

F
T
F
F
T

3. Let V be a vector space and let W, Z be subspaces. Prove that $W \cap Z$ is a subspace. (4)

To show that $W \cap Z$ is a subspace, we must show that (a) it contains 0, (b) it is closed under addition, and (c) it is closed under multiplication. *(1 mark)*

(a): $0 \in W$ and $0 \in Z$; therefore $0 \in W \cap Z$. *(1 mark)*

(b): Let $v, w \in W \cap Z$. Then $v, w \in W$ and since W is a subspace, $v + w \in W$. Likewise, $v + w \in Z$. Hence, $v + w \in W \cap Z$. *(1 mark)*

(c): Let $v \in W \cap Z$ and $c \in F$. Then $v \in W$ and since W is a subspace, $cv \in W$. Likewise, $cv \in Z$. Hence $cv \in C \cap Z$. *(1 mark)*

4. Determine whether the following maps $S, T : \mathbb{R}[t] \to \mathbb{R}^2$ are linear. Justify your answer.

(a) $S(p(t)) = (p(0) + p(1), p'(0))$.

(b) $T(p(t)) = (\cos(p(0)), \sin(p(0))$. (4)

(a): S is linear. *(1 mark)*

To see this, let $p, q \in \mathbb{R}[t]$ and let $c \in \mathbb{R}$. Then

$$\begin{aligned} S(p+q) &= ((p+q)(0) + (p+q)(1), (p+q)'(0)) \\ &= (p(0) + q(0) + p(1) + q(1), p'(0) + q'(0)) \\ &= (p(0) + p(1), p'(0)) + (q(0) + q(1), q'(0)) \\ &= S(p) + S(q). \end{aligned}$$

Similarly,

$$\begin{aligned} S(cp) &= ((cp)(0) + (cp)(1), (cp)'(0)) \\ &= (cp(0) + cp(1), cp'(0)) \\ &= c(p(0) + p(1), p'(0)) \\ &= cS(p). \end{aligned}$$

(1 mark)

(b): T is not linear. *(1 mark)*

For example, let $p(t) = \pi$ and let $c = 2$. Then

$$T(cp(t)) = (\cos(2\pi), \sin(2\pi)) = (1, 0),$$

and

$$cT(p(t)) = 2(\cos(\pi), \sin(\pi)) = 2(-1, 0) = (-2, 0).$$

Since these aren't equal, T can't be linear. *(1 mark)*

5. Let V be a finite dimensional vector space, let $T : V \to V$ be a linear map, and let $v \in V$. Suppose that $T(v)$ is nonzero. Prove that there exist ordered bases α, β of V such that the first column of $[T]_\alpha^\beta$ is

$$\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

(4)

Since $T(v)$ is nonzero, v is nonzero. *(1 mark)*

Hence the sets $\{v\}$ and $\{T(v)\}$ are linearly independent. *(0.5 marks)*

So they may be extended to bases $\alpha = \{v_1, \ldots, v_n\}$ and $\beta = \{w_1, \ldots, w_n\}$ respectively; that is, $v_1 = v$ and $w_1 = T(v)$. *(1 mark)*

By definition, the first column of $[T]_\alpha^\beta$ is

$$\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix},$$

where $T(v_1) = a_{11}w_1 + \cdots + a_{n1}w_n$. *(0.5 marks)*

Since

$$T(v_1) = T(v) = w_1,$$

we see that $a_{11} = 1$ and $a_{21} = \cdots = a_{n1} = 0$, as required. *(1 mark)*

6. Let V, W be vector spaces and let $T, U : V \to W$ be nonzero linear maps. Suppose that $\ker(T) \neq \ker(U)$. Prove that $\{T, U\}$ is a linearly independent set in $\mathcal{L}(V, W)$. (3)

Suppose that $aT + bU = 0$ in $\mathcal{L}(V, W)$, for some $a, b \in F$. We need to show that $a = b = 0$. *(1 mark)*

For a contradiction, suppose that it isn't the case that $a = b = 0$. WLOG, $a \neq 0$, and then $T = -a^{-1}bU$; that is, T is a scalar multiple of U. Let us write $c = -a^{-1}b$, so that $T = cU$.

Then for $v \in \ker(U)$, we have

$$T(v) = cU(v) = c \cdot 0 = 0.$$

This shows that $\ker(U) \subseteq \ker(T)$. *(1 mark)*

Since T is nonzero, we must have $c \neq 0$ so that U is a scalar multiple of T. Hence by the same argument, $\ker(T) \subseteq \ker(U)$. Therefore, $\ker(T) = \ker(U)$, a contradiction. *(1 mark)*

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