Lesson 1: Introduction and Exponentials

Introduction

Welcome one and all! I hope you are as excited as I am. Over the next 8 months we will be journeying into the world of higher mathematics, going leaps and bounds further than most ever will. But first, we have to go over some administrative (read: boring) stuff, but don't worry, in this lesson we will both see an overview of exponentials and see our first ever proofs of mathematical statements.

Talking points in the Course Outline:

- I almost forgot to introduce myself! My name is Zac Zanussi, and I am a Master's of Science candidate here at Carleton University, specializing in Pure Mathematics. My research topics include quantum information theory, the Heisenberg group and it's representations, and quantum channels. These terms may seem pretty intimidating, but they have their foundations firmly rooted in topics that we will study in this course, including modular arithmetic and finite fields, complex numbers, and of course, trig. Hopefully I will be able to show you a little bit of my research topic in lesson 15. But I'm getting ahead of myself.

- You can see my email at the top of the page. Please do not hesitate to contact me with questions throughout the semester. This includes questions that you may have about assignments or concepts that we are studying.

- Below that is the course webpage. Here I will be posting all assignments, as well as my (meticulously typesetted) lecture notes. Please take advantage of this resource, and check it often. I will also post additional problems, solutions, and other miscellaneous goodies. For example, posted now is a list of some of my favourite YouTube channels that discuss math topics.

- Next is a list of the topics we will cover in each term.

- Lectures are every Thursday at this time. There are a few missed weeks; there will be NO class on October 24^{th} or February 20^{th}. These are the university's reading weeks, and these are the times that I am madly scrambling to catch up on work and sleep. There is also a week off in the end of January (date TBD) so that you can focus on your high school exams. We will vote on which week to take off in the beginning of January.

- Notice that I said "lecture"; this course is designed to emulate an honours mathematics course that you may take in a university like Carleton. As such, classes will be given in a lecture format, where you will be expected to take notes. In this environment, the onus is on YOU to keep up with the course work, and to ask questions or get help if you are struggling. But don't worry! I'll be there to help you along the way. I encourage you to ask questions during lectures, and I am holding a 30 minute office hour after class where you can ask any questions you may have.

- Now let's talk about evaluation. There will be weekly assignments (you will get the first at the end of class) that you are expected to complete. They will be due two weeks later, at the beginning of lecture. This gives you plenty of time to digest the material, especially during weeks where you are busy in the other aspects of your life. However, make sure that you aren't leaving them to the last day; studies show that material is absorbed best if you are exposed to it over time. That is, it's better to study an hour a day for 7 days than it is to study 7 hours for one day. For this reason, I recommend that you block off a chuck of time every day to study and work on the assignments. It doesn't have to be long; as little as 30 minutes a day might be sufficient if you do it every day!

- At the end of each term there will be a test, which is taken during the usual lecture time. The dates for these are on the Course Outline. Please let me know as soon as possible if you have a scheduling conflict, so that we can make alternate arrangements.

- The cut off to pass the course is 50%. You can see the mark breakdown. If you pass, you will receive a nice certificate and become eligible for the "*Mathematics For Success*" scholarship. This is a $400 award for anyone enrolled in a program under the Faculty of Mathematics and Statistics (or Double Major in Physics and Math) here at Carleton University. Please see me for more information.

That's done with the Course Outline, but we've still got a few more talky bits to cover;

- Mathematics in university is not only more difficult than the stuff you are used to; it's also fundamentally different in a big way. In classes you are used to, you are introduced to a particular problem, given the method to solve it, then asked to solve very similar problems over and over until you have memorized the process. That is not how we will do things. Upper level maths is more concerned with *proving* statements in general, rather than solving individual problems. For example, rather than finding the roots of a particular polynomial, we are more concerned with determining necessary and sufficient conditions for the existence of roots. By abstracting a problem and solving it in general, we can solve a whole class of problems at once, and then leave the details to the engineers or physicists or chemists who will be using our work. Here is a more concrete example; perhaps you are doing some calculations, and you notice that when you multiply two even numbers, say 4 and 12, you get another even number, 48. You notice this a number of times and you decide to investigate. After trying many different combinations, you want to conclude that the product of two even numbers is always even. But can you be sure? What if all the combinations you tried are simply coincidence, and there are two even numbers you didn't try whose product is odd. There are infinitely many numbers, so you can't just try them all. In this lecture, we will *prove* that the product of even numbers is even. We will use this result to prove a more interesting one; that the square root of 2 is irrational.

- Do your assignments! This is now the fifth year that I will be teaching this course, and the students who do the assignments and study early and often are always the most successful. This includes reviewing the material often, especially in the days following a lecture. Remember, a little bit of work over time is better than a lot right at the end.

- Registration for the course will stay open until after next week's lecture; if you have a friend who might like this course, invite them next week! It's always easier to stay motivated when you bring a friend along (and a full class is a fun class).

Wow! More than two pages and we haven't even gotten to the good stuff. Here goes.

Notation

We start by reviewing some basic notation that will follow us for the whole course. Most of this stuff is universal across all of mathematics, so we might as well nail it down now.

Sets of numbers: these are familiar sets that we will use often, so it makes sense to have an agreed upon notation. The way I write them, both on the board and in this document, is called *blackboard bold.*

$\mathbb{N}$: The natural numbers. These are the counting numbers, starting with 1 and 2 and going off to infinity. Note that many authors don't include 0 in this set (some do). I usually do not, but if I want to include 0 I will specify explicitly.

$\mathbb{Z}$: The integers. These are the naturals, the negative naturals, and 0. This set goes of to infinity in one direction and negative infinity in the other. We can write the set like this (the := means "is defined to be");

$$\mathbb{Z} := \{\dots, -2, -1, 0, 1, 2, \dots\}.$$

$\mathbb{Q}$: The rational numbers. These are the set of all numbers that can be written as a fraction with an integer on the top and the bottom. So $\mathbb{Q}$ contains the integers, as well as $\frac{1}{2}$, $\frac{4}{17}$, $\frac{45245}{92239}$, etc. We can write this in the following way;

$$\mathbb{Q} := \{\frac{a}{b} | a, b \in \mathbb{Z}, b > 0\}.$$

One reads this as "$\mathbb{Q}$ is defined to be the set of all fractions a over b, such that a and b are integers, and b is greater than 0." We ask that b is positive simply for convenience, note that removing this restriction doesn't change our definition of the rationals (of course, b can't be 0 either way).

$\mathbb{R}$: The real numbers. "What?" I hear some of you thinking, "Does that mean there are numbers that aren't real?" Well, yes, there are *imaginary* numbers, which we will study in the winter semester. Don't get too hung up on the names. The real numbers are the rationals numbers, as well as irrational numbers (numbers that can't be written as a fraction of integers). These include numbers such as π, e (Euler's number), and $\sqrt{2}$, and basically any other number that you can think of. There are a number of different ways to write the reals as a set, but they are beyond our scope so we will leave this as it is.

These are the sets that we will be mostly concerned with, until later when we will introduce complex/imaginary numbers.

$\in$: "is an element of." For example, $a \in \mathbb{Z}$ is just a shorthand way to say "let a be an integer."

Introduction to Exponentials

We are now finally ready to talk about exponentials. I know that some of you will have studied this before, but we are hopefully going to tackle it in a way that you haven't seen.

Recall: Multiplication is simply repeated addition. For example,

$$5 \cdot 3 = 5 + 5 + 5 = 3 + 3 + 3 + 3 + 3 = 3 \cdot 5$$

More generally, we can write

$$n \cdot m = n + n + \cdots + n = m + m + \cdots + m = m \cdot n$$

In the same way, *exponentiation* is just repeated multiplication;

Definition 1. For $a, n \in \mathbb{R}$, $a \geq 0$, the statement a^n, read "a to the exponent n", is defined to be a multiplied by itself n times. That is,

$$a^n = a \cdot a \cdot \cdots \cdot a.$$

We call a the *base* and n the *exponent*. Some people call the expression a^n a power.

So we can think of exponentiation as simply a shorthand for many multiplications. This can help us intuitively grasp what an equation involving exponentials is trying to say. Here's an example;

$$3^5 = 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = 243.$$

Note that unlike multiplication, we can't simply change the order of the numbers being multiplied. That is, $3^5 \neq 5^3$. We say that multiplication is *commutative* while exponentiation is not.

With just this definition, we can actually prove all of the exponential rules that you learn in class.

Identities

Here we will prove many of the exponential identities that you may already be familiar with. At this point, the only tools we can use are the definition of exponentials and whatever else we prove along the way. Let $a \in \mathbb{R}$, $a \geq 0$, and let $x, y \in \mathbb{R}$.

- $a^x \cdot a^y = a^{x+y}$

  Proof.

  $$a^x \cdot a^y = a \cdot a \cdot \cdots \cdot a \cdot a \cdot \cdots \cdot a = a^{x+y}$$

  □

  That is, *the product of powers is the sum of exponents.*

- $\dfrac{a^x}{a^y} = a^{x-y}$

Proof.

$$\frac{a^x}{a^y} = \frac{a \cdot \dots \cdot a}{a \cdot \dots \cdot a} = a^{x-y}$$

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That is, *the quotient of powers is the difference of exponents.*

- $(a^x)^y = a^{x \cdot y}$ The proof of this one is on the assignment. Make sure your proof looks like the ones I've given. By the way, *the power of a power is the product of exponents.*
- Let $b \in \mathbb{R}$, $b > 0$. Then $(ab)^x = a^x b^x$.

  Proof.

  $$(ab)^x = ab \cdot \dots \cdot ab = a \cdot \dots \cdot ab \cdot \dots \cdot b = a^x b^x$$

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  That is, *the power of a product is the product of powers.*
- If $a \neq 0$, then $a^0 = 1$.

  Proof. Let $n \in \mathbb{N}$. Then

  $$a^0 = a^{(n-n)} = \frac{a \cdot \dots \cdot a}{a \cdot \dots \cdot a} = 1$$

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  That is, *anything (except 0) to the power of 0 is 1.* Why can't $a = 0$?
- What is 0^0? There's a problem; by above, we would like anything to the power of 0 to be 1, so $0^0 = 1$. But by the definition of exponentials, 0^0 is 0 multiplied by itself, and anything times 0 is 0, so $0^0 = 0$. So since we can't come to agreement, we choose not to define it, and thus we can never make sense of such an expression.

What happens if we have a negative exponent? We define

$$a^{-n} = \frac{1}{a^n}.$$

Also, it is sometimes convenient to use *radicals*;

$$a^{\frac{1}{n}} = \sqrt[n]{a}.$$

We also had the requirement above that $a \geq 0$. Why is that? Well,

$$(-a)^2 = (-a)(-a) = (-1)^2 a^2 = a^2 \neq -a^2.$$

$$(-a)^3 = (-1)^3 a^3 = -a^3 \neq a^3.$$

So in general, we prefer to keep the bases positive, since we can always simplify in this way.

Example 1. Let's do some practice exponentials with actual numbers;

$5^2 = 5 \cdot 5 = 25$ $\quad$ $1^1 = 1^2 = 1^3 = 1$

$5^{-2} = \frac{1}{5 \cdot 5} = \frac{1}{25} = 0.04$ $\quad$ $0.2^3 = 0.2 \cdot 0.2 \cdot 0.2 = 0.008$

$5^0 = 1$ $\quad$ $4^{3.5} = 4 \cdot 4 \cdot 4 \cdot 4^{\frac{1}{2}} = 4 \cdot 4 \cdot 4 \cdot 2 = 128$

It is useful to understand how equivalent powers may be related. Let $a, b, x, y \in \mathbb{R}$ with $a, b \geq 0$. Then

$$a^x = a^y \iff x = y.$$

That double arrow means that the two statements are *equivalent*, that is, one implies the other. We read the arrow as "if and only if". So if powers are equal then the bases are equal if and only if the exponents are equal.

Example 2. Find k such that $2\sqrt{2} = 8^k$.

Let's make the bases equal, then we can use the fact above.

$$\begin{aligned} 2\sqrt{2} &= 8^k \\ 2^{\frac{2}{2}} \cdot 2^{\frac{1}{2}} &= \\ 2^{\frac{3}{2}} &= \\ (2^3)^{\frac{1}{2}} &= \\ 8^{\frac{1}{2}} &= 8^k \end{aligned}$$

So $k = \frac{1}{2}$.

Similarly,

$$a^x = b^x \iff a = b,$$

or, if the powers are equal then exponents are equal if and only if the bases are equal.

Example 3. Find n such that $2^{30} = n^{10}$.

Let's make the exponents equal, then we can use the fact above.

$$2^{30} = (2^3)^{10} = n^{10}.$$

So by the fact above, we must have

$$2^3 = n$$

and thus $n = 8$.

There is one base that is special. This is $e \approx 2.718$, called *Euler's number* (pronounced "Oiler", like the hockey team). This number appears in calculus, algebra, and basically everywhere else. It is irrational, like π. Sometimes we speak of the *exponential function*; by this we usually mean

$$\exp(x) = e^x.$$

Example 4. Find n such that $\dfrac{2^{-3}}{4^{n+5}} = 8^{2n-1}$.

Perhaps you have noticed that if $0 < a < 1$ and $x > 1$, then $a > a^x$. We can also see that if $a > 1$ and $x > 1$, $a < a^x$. In words, if your base is small (ie $0 < a < 1$) then applying a big exponent makes the value smaller, while if your base is big (ie $a > 1$) applying a big exponent makes the value bigger. We will use these facts to help us in the next example.

Example 5. Which is larger; $\dfrac{9^9}{10^9}$ or $\dfrac{9^{10}}{10^{10}}$?

Example 6. Simplify $2^{22} - 2^{21}$.

Introduction to Higher Mathematics

Mathematicians have very different jobs than most people assume. We are not just multiplying bigger numbers, or doing tougher integrals. In fact, as you get deeper into pure mathematics, you may rarely do computations at all. In my research, the only numbers I come across with regularity are 0,1,π, e and sometimes maybe a 2.

What mathematicians prefer to do is *prove* things. That is, we make statements about whatever we are working with, and we either show that it is always true by proving it, or show that it is not true by finding a counterexample where it is not true. We tend to follow a set process:

First, we make *axioms.* This is like setting the rules of the game we want to play. An axiom is a fact that is considered so basic that we simply assume it to be true. In basic arithmetic (ie, the mathematics of computation in the real numbers), axioms include, for example,

$$\text{for any } x \in \mathbb{R},\ x + 0 = x.$$

Could you imagine trying to prove such a thing? So we set it to be a rule that we assume is true. Another example of an axiom comes from geometry; "two parallel lines never intersect". It turns out that it is impossible to prove this using our other basic geometry rules, and yet we want it to be true, so we assume it from the start. We can change a lot about the math that we do by switching up the axioms (for example, Non-Euclidean Geometry doesn't use this axiom, and it has some very surprising results because of it).

Next, we make *definitions.* These are convenient ways to denote certain phenomena so that we can talk about them. For example, in arithmetic we define *odd* and *even* numbers as such;

Definition 2. A number $a \in \mathbb{Z}$ is called *even* if there exists $k \in \mathbb{Z}$ such that

$$a = 2k.$$

A number $o \in \mathbb{Z}$ is called *odd* if there exists $k \in \mathbb{Z}$ such that

$$o = 2k + 1.$$

Now we can talk about even and odd numbers without risk of confusion. Mathematicians always agree on rigorous definitions before they start working.

Next, we begin working with our definitions, finding relations, and proving *lemmas* and *propositions.* These are elementary results that we can prove in a blackboard or two. They help us prove the bigger statements later by breaking the workload down into smaller chunks. We can prove a lemma right now;

Lemma 1. *The product of two odd numbers must be odd.*

Proof. Let $a, b \in \mathbb{Z}$ be odd numbers. By definition, that means that there exists $n, m \in Z$ such that $a = 2n + 1$ and $b = 2m + 1$. Then, multiplying the two, we have

$$a \cdot b = (2n + 1) \cdot (2m + 1) = 4mn + 2n + 2m + 1 = 2(2mn + n + n) + 1.$$

Now set $k = 2mn + n + m$; it is obvious that $k \in \mathbb{Z}$. Thus

$$a \cdot b = 2k + 1,$$

which tells us that $a \cdot b$ is odd, by definition. $\square$

We use this very basic result to prove a less obvious one;

Proposition 1. *Let $a \in \mathbb{Z}$. If a^2 is even, then a is also even.*

Proof. Note that we can write $a^2 = a \cdot a$. If a were odd, then by the lemma we would have that a^2 is odd. But it's not odd, by assumption, so we must have that a is even. $\square$

Now we have the final step - *theorems*. These are the mathematicians bread and butter. A theorem is a mathematical result that is either very useful or very interesting; often both! Most of a mathematicians job involves proving theorems. This is useful because once a theorem is proved, then any other mathematician, physicist, engineer, chemist, high school teacher, accountant, etc can use the theorem and be 100% confident that it is true and reliable. Let's prove our first theorem;

Theorem 1. *The number $\sqrt{2}$ is irrational; that is, there do not exist integers a, b such that $\sqrt{2} = \frac{a}{b}$.*

Proof. We do this proof by *contradiction*; that is, we assume the opposite of what we are trying to show, and use that to derive some impossibility, or contradiction. That proves that the opposite of what we want can't be true, which proves that what we want must we true. In this case, it means assuming that there exists $a, b \in \mathbb{Z}$ such that

$$\sqrt{2} = \frac{a}{b}.$$

We add a few more assumptions to make the proof work; namely, we ask that $\frac{a}{b}$ is in lowest terms. This is a reasonable request, as if they are not, we just reduce them so they are. Note that, in particular, this means that a and b are not *both* even. Now, we observe that

$$\sqrt{2} = \frac{a}{b} \;\Rightarrow\; 2 = \frac{a^2}{b^2} \;\Rightarrow\; 2b^2 = a^2.$$

Since $b^2 \in \mathbb{Z}$, we have that a^2 is even! But by our proposition, we must also have that a is even. So we can find $k \in \mathbb{Z}$ such that $a = 2k$. Subbing this into the equation above, we get

$$a^2 = (2k)^2 = 4k^2 = 2b^2. \;\Rightarrow\; b^2 = 2k^2.$$

Now this tells us that b^2 is even! Again by our proposition, b must be even.

Wait a minute; didn't we assume at the beginning that a and b *weren't* both even? That's a contradiction! So it must be true that $\sqrt{2}$ is irrational, because if it's not, then we break rules that we had already set. $\square$

That's the end of lesson 1. Get working on your assignment (start early!) and I'll see you next week to study logarithms.