PHYS 304 Final Exam
Tuesday, December 13, 2011

(Open book exam. No computers/phones/ipods/ipads... are allowed)

Problem 1. Consider a 2p state of the Hydrogen atom, expressed in $|n,l,m\rangle$ notations as $|2,1,1\rangle$.

What is each of the following (see below) in $|n,l,m\rangle$ notation? Formulate your answer also in coordinate representation (in terms of $R_{nl}(r)$ and $Y_l^m(\theta,\phi)$ functions). You do not have to write down the actual functional form of the wave functions.

(**a**) $L_x|2,1,1\rangle$
(**b**) $L_y|2,1,1\rangle$
(**c**) $L_z|2,1,1\rangle$
(**d**) $L_x^2|2,1,1\rangle$
(**e**) $L_y^2|2,1,1\rangle$

Hint: use raising and lowering operators to solve this.

Problem 2. Consider the normalized spin state written in spin -z basis as follows:

$$|\chi\rangle = \frac{1}{\sqrt{10}}\begin{pmatrix}3\\ i\end{pmatrix},$$

(**a**) Is this state $|\chi\rangle$ an eigenstate of $\hat{S}^2$? Is it eigenstate of $\hat{S}_z$? Is it eigenstate of $\hat{S}_x$? Is it eigenstate of $\hat{S}_y$? (Justify your answers). In each case, if it is an eigenstate, give the eigenvalue.

(**b**) If the spin state is as given above, and a measurement is made of the z component, what are the possible results of that measurements and what are the probabilities of each possible result?

(**c**) If the spin state is as given above, what is the expectation value of $\hat{S}_y$?

(**d**) If the spin state is as given above, what is the expectation value of $\hat{S}_z$?

(**e**) In problem d) you have computed the expectation value $\hat{S}_z$. In problem b) you have computed the probabilities to find spin up or down along z axis. Is there any relation between these numbers?

Problem 3. Consider a particle confined in a **one dimensional harmonic potential**, $V(x) = \frac{1}{2}m\omega^2x^2$. Consider the state

$$|\psi\rangle = A\left(\hat{a}\hat{a}^\dagger + \hat{a}\hat{a}^\dagger\hat{a}^\dagger\right)|0\rangle,$$

where $\hat{a} = \frac{1}{\sqrt{2\hbar m\omega}}(i\hat{p} + m\omega\hat{x})$ and $\hat{a}^\dagger = \frac{1}{\sqrt{2\hbar m\omega}}(-i\hat{p} + m\omega\hat{x})$ is the hermitian conjugate of $\hat{a}$ and $|0\rangle = (\frac{m\omega}{\pi\hbar})^{1/4}e^{-\frac{m\omega x^2}{2\hbar}}$ is the ground state with energy $E_0 = \frac{1}{2}\hbar\omega$.

(**a**) What is the normalization constant A? Represent state $|\psi\rangle$ in the Dirac's notations as a superposition of $|n\rangle$ states, where $|n\rangle$ satisfies $\hat{H}|n\rangle = \hbar\omega\left(n+\frac{1}{2}\right)|n\rangle$.

(**b**) Consider two new states $|\psi_1\rangle \sim \hat{a}\hat{a}^\dagger|\psi\rangle$ and $|\psi_2\rangle \sim \hat{a}^\dagger\hat{a}|\psi\rangle$. What these states are in the Dirac's notations $|n\rangle$? Normalize these new states $|\psi_1\rangle$ and $|\psi_2\rangle$ such that $\langle\psi_1|\psi_1\rangle = 1$ and $\langle\psi_2|\psi_2\rangle = 1$.

(**c**) What is the expectation value of the total energy $\langle\psi_1|H|\psi_1\rangle$?

(**d**) What is the expectation value of the potential energy $\langle\psi_1|V|\psi_1\rangle$?

(**e**) Write the time-dependent wave function $|\psi_1(t)\rangle$ and, using this, compute the expectation value $\langle\psi_1(t)|\hat{x}|\psi_1(t)\rangle$.

Problem 4. The electron in a hydrogen atom is described by the following superposition of two states:

$$|\psi\rangle \equiv \frac{1}{\sqrt{2}}\Big(|n=2,l=1,m=0,s_z=+1/2\rangle + |n=2,l=0,m=0,s_z=+1/2\rangle\Big)$$

(**a**) What are the expectation values of the following operators:

$$\langle\psi|L^2|\psi\rangle,\ \langle\psi|L_z|\psi\rangle,\ \langle\psi|S^2|\psi\rangle,\ \langle\psi|S_z|\psi\rangle.$$

(**b**) Let $\mathbf{J} = \mathbf{L} + \mathbf{S}$ be the total angular momentum. Express state $|\psi\rangle$ in basis $|n,l,J,J_z\rangle$.

Hint: use the Clebsch -Gordan coefficients from the table on pg. 188

(**c**) If you measured J^2, what values might you get, and what is the probability of each? Do the same for J_z.

(**d**) What are the expectation values of the following operators:

$$\langle\psi|J^2|\psi\rangle,\ \langle\psi|J_z|\psi\rangle.$$

(**e**) Express $|\psi\rangle$ in coordinate representation using $R_{nl}(r)$ and $Y_l^m(\theta,\phi)$ functions and compute the expectation value $\langle\psi|z|\psi\rangle$. *(this expectation value determines the behaviour of a hydrogen atom when it is placed in a uniform external electric field E_0 as the electric potential in this case is proportional to $V \sim E_0 z$. It is the so called Stark effect.).*

Hint: a simple way to compute integrals of the form $\int_0^\infty dr r^n \exp(-\alpha r)$ is to differentiate formula $\int_0^\infty dr \exp(-\alpha r) = \alpha^{-1}$ n-times with respect to α.