ENGR-213 RECOMMENDED PROBLEMS (WEEK 1-6)

1.1 Definition and Terminology: 1,3,5,6,8,10,11,13,14,21,23

In Problems 1–8, state the order of the given ordinary differential equation. Determine whether the equation is linear or nonlinear by matching it with (6).

1. $(1 - x)y'' - 4xy' + 5y = \cos x$

3. $t^5y^{(4)} - t^3y'' + 6y = 0$

5. $\dfrac{d^2y}{dx^2} = \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}$

6. $\dfrac{d^2R}{dt^2} = -\dfrac{k}{R^2}$

8. $\ddot{x} - (1 - \frac{1}{3}\dot{x}^2)\dot{x} + x = 0$

In Problems 9 and 10, determine whether the given first-order differential equation is linear in the indicated dependent variable by matching it with the first differential equation given in (7).

10. $u\,dv + (v + uv - ue^u)\,du = 0$; in v; in u

In Problems 11–14, verify that the indicated function is an explicit solution of the given differential equation. Assume an appropriate interval I of definition for each solution.

11. $2y' + y = 0; \quad y = e^{-x/2}$

13. $y'' - 6y' + 13y = 0; \quad y = e^{3x}\cos 2x$

14. $y'' + y = \tan x; \quad y = -(\cos x)\ln(\sec x + \tan x)$

In Problems 21–24, verify that the indicated family of functions is a solution of the given differential equation. Assume an appropriate interval I of definition for each solution.

21. $\dfrac{dP}{dt} = P(1 - P); \quad P = \dfrac{c_1e^t}{1 + c_1e^t}$

23. $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 4y = 0; \quad y = c_1e^{2x} + c_2xe^{2x}$

1.2 Initial Value Problems: 7,9,11,12,17,18

In Problems 7–10, $x = c_1\cos t + c_2\sin t$ is a two-parameter family of solutions of the second-order DE $x'' + x = 0$. Find a solution of the second-order IVP consisting of this differential equation and the given initial conditions.

7. $x(0) = -1, \quad x'(0) = 8$

9. $x(\pi/6) = \frac{1}{2}, \quad x'(\pi/6) = 0$

In Problems 11–14, $y = c_1e^x + c_2e^{-x}$ is a two-parameter family of solutions of the second-order DE $y'' - y = 0$. Find a solution of the second-order IVP consisting of this differential equation and the given initial conditions.

11. $y(0) = 1, \quad y'(0) = 2$

12. $y(1) = 0, \quad y'(1) = e$

In Problems 17–24, determine a region of the xy-plane for which the given differential equation would have a unique solution whose graph passes through a point (x_0, y_0) in the region.

17. $\dfrac{dy}{dx} = y^{2/3}$

18. $\dfrac{dy}{dx} = \sqrt{xy}$

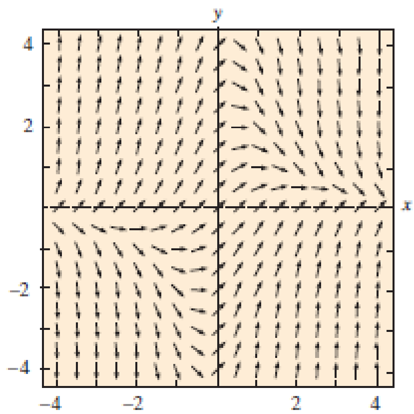
2.1 Solution curves without a solution: 3, 4, 26, 27

In Problems 1–4, reproduce the given computer-generated direction field. Then sketch, by hand, an approximate solution curve that passes through each of the indicated points. Use different colored pencils for each solution curve.

3. $\dfrac{dy}{dx} = 1 - xy$

(a) $y(0) = 0$ **(b)** $y(-1) = 0$

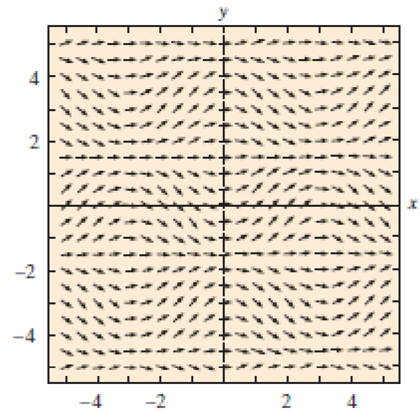
(c) $y(2) = 2$ **(d)** $y(0) = -4$



4. $\dfrac{dy}{dx} = (\sin x)\cos y$

(a) $y(0) = 1$ **(b)** $y(1) = 0$

(c) $y(3) = 3$ **(d)** $y(0) = -\frac{5}{2}$



In Problems 21–28, find the critical points and phase portrait of the given autonomous first-order differential equation. Classify each critical point as asymptotically stable, unstable, or semi-stable. By hand, sketch typical solution curves in the regions in the xy-plane determined by the graphs of the equilibrium solutions.

26. $\dfrac{dy}{dx} = y(2 - y)(4 - y)$

27. $\dfrac{dy}{dx} = y\ln(y + 2)$

2.2 Separable Equations: 7,9,13,19,25,27

In Problems 1–22, solve the given differential equation by separation of variables.

7. $\dfrac{dy}{dx} = e^{3x+2y}$

9. $y\ln x\dfrac{dx}{dy} = \left(\dfrac{y+1}{x}\right)^2$

13. $(e^y + 1)^2 e^{-y}dx + (e^x + 1)^3 e^{-x}dy = 0$

19. $\dfrac{dy}{dx} = \dfrac{xy + 3x - y - 3}{xy - 2x + 4y - 8}$

In Problems 23–28, find an implicit and an explicit solution of the given initial-value problem.

25. $x^2\dfrac{dy}{dx} = y - xy, \quad y(-1) = -1$

27. $\sqrt{1 - y^2}\,dx - \sqrt{1 - x^2}\,dy = 0, \quad y(0) = \sqrt{3}/2$

2.3 Linear Equations: 7,9,23,27,31

In Problems 1–24, find the general solution of the given differential equation. Give the largest interval over which the general solution is defined. Determine whether there are any transient terms in the general solution.

7. $x^2y' + xy = 1$

9. $x\dfrac{dy}{dx} - y = x^2 \sin x$

23. $x\dfrac{dy}{dx} + (3x + 1)y = e^{-3x}$

In Problems 25–32, solve the given initial-value problem. Give the largest interval I over which the solution is defined.

27. $L\dfrac{di}{dt} + Ri = E; \quad i(0) = i_0$, L, R, E, and i_0 constants

31. $\left(\dfrac{e^{-2\sqrt{x}} - y}{\sqrt{x}}\right)\dfrac{dx}{dy} = 1, \quad y(1) = 1$

2.4 Exact Equations, integrating factors: 3,5,9,15,27,29,31

In Problems 1–20, determine whether the given differential equation is exact. If it is exact, solve it.

3. $(5x + 4y)\,dx + (4x - 8y^3)\,dy = 0$

5. $(2xy^2 - 3)\,dx + (2x^2y + 4)\,dy = 0$

9. $(x - y^3 + y^2 \sin x)\,dx = (3xy^2 + 2y \cos x)\,dy$

15. $\left(x^2y^3 - \dfrac{1}{1 + 9x^2}\right)\dfrac{dx}{dy} + x^3y^2 = 0$

In Problems 27 and 28, find the value of k so that the given differential equation is exact.

27. $(y^3 + kxy^4 - 2x)\,dx + (3xy^2 + 20x^2y^3)\,dy = 0$

In Problems 29 and 30, verify that the given differential equation is not exact. Multiply the given differential equation by the indicated integrating factor $\mu(x, y)$ and verify that the new equation is exact. Solve.

29. $(-xy \sin x + 2y \cos x)\,dx + 2x \cos x\,dy = 0; \quad \mu(x, y) = xy$

In Problems 31–36, solve the given differential equation by finding, as in Example 4, an appropriate integrating factor.

31. $(2y^2 + 3x)\,dx + 2xy\,dy = 0$

2.5 Solutions by Substitution (Bernoulli, homogeneous, linear substitution): 5,7,9,13,17,19,21,25,27

In Problems 1–10, solve the given differential equation by using an appropriate substitution.

5. $(y^2 + yx)\,dx - x^2\,dy = 0$

7. $\dfrac{dy}{dx} = \dfrac{y - x}{y + x}$

9. $-y\,dx + (x + \sqrt{xy})\,dy = 0$

In Problems 11–14, solve the given initial-value problem.

13. $(x + ye^{y/x})\,dx - xe^{y/x}\,dy = 0, \quad y(1) = 0$

Each DE in Problems 15–22 is a Bernoulli equation.

In Problems 15–20, solve the given differential equation by using an appropriate substitution.

17. $\dfrac{dy}{dx} = y(xy^3 - 1)$

19. $t^2\dfrac{dy}{dt} + y^2 = ty$

In Problems 21 and 22, solve the given initial-value problem.

21. $x^2\dfrac{dy}{dx} - 2xy = 3y^4, \quad y(1) = \dfrac{1}{2}$

Each DE in Problems 23–30 is of the form $\dfrac{dy}{dx} = f(Ax + By + C)$

In Problems 23–28, solve the given differential equation by using an appropriate substitution.

25. $\dfrac{dy}{dx} = \tan^2(x + y)$

27. $\dfrac{dy}{dx} = 2 + \sqrt{y - 2x + 3}$

1.3 Differential Equations as Mathematical Models: 1,2,3,5,7,9,1013,15,16,19

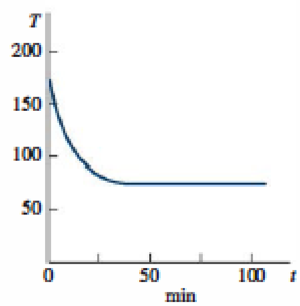
≡ Population Dynamics

1. Under the same assumptions underlying the model in (1), determine a differential equation governing the growing population $P(t)$ of a country when individuals are allowed to immigrate into the country at a constant rate $r > 0$. What is the differential equation for the population $P(t)$ of the country when individuals are allowed to emigrate at a constant rate $r > 0$?
2. The population model given in (1) fails to take death into consideration; the growth rate equals the birth rate. In another model of a changing population of a community, it is assumed that the rate at which the population changes is a *net* rate—that is, the difference between the rate of births and the rate of deaths in the community. Determine a model for the population $P(t)$ if both the birth rate and the death rate are proportional to the population present at time t.
3. Using the concept of a net rate introduced in Problem 2, determine a differential equation governing a population $P(t)$ if the birth rate is proportional to the population present at time t but the death rate is proportional to the square of the population present at time t.

≡ Newton's Law of Cooling/Warming

5. A cup of coffee cools according to Newton's law of cooling (3). Use data from the graph of the temperature $T(t)$ in **FIGURE 1.3.10** to estimate the constants T_m, T_0, and k in a model of the form of the first-order initial-value problem

$$\frac{dT}{dt} = k(T - T_m), \quad T(0) = T_0.$$



≡ Spread of a Disease/Technology

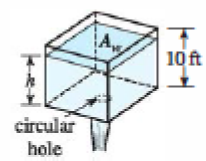
7. Suppose a student carrying a flu virus returns to an isolated college campus of 1000 students. Determine a differential equation governing the number of students $x(t)$ who have contracted the flu if the rate at which the disease spreads is proportional to the number of interactions between the number of students with the flu and the number of students who have not yet been exposed to it.

≡ Mixtures

9. Suppose that a large mixing tank initially holds 300 gallons of water in which 50 pounds of salt has been dissolved. Pure water is pumped into the tank at a rate of 3 gal/min, and when the solution is well stirred, it is pumped out at the same rate. Determine a differential equation for the amount $A(t)$ of salt in the tank at time t. What is $A(0)$?
10. Suppose that a large mixing tank initially holds 300 gallons of water in which 50 pounds of salt has been dissolved. Another brine solution is pumped into the tank at a rate of 3 gal/min, and when the solution is well stirred, it is pumped out at a *slower* rate of 2 gal/min. If the concentration of the solution entering is 2 lb/gal, determine a differential equation for the amount $A(t)$ of salt in the tank at time t.

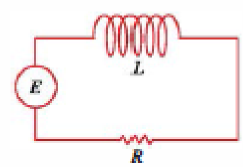
≡ Draining a Tank

13. Suppose water is leaking from a tank through a circular hole of area A_h at its bottom. When water leaks through a hole, friction and contraction of the stream near the hole reduce the volume of the water leaving the tank per second to $cA_h\sqrt{2gh}$, where c $(0 < c < 1)$ is an empirical constant. Determine a differential equation for the height h of water at time t for the cubical tank in **FIGURE 1.3.12**. The radius of the hole is 2 in and $g = 32$ ft/s^2.

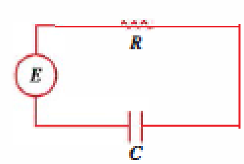


≡ Series Circuits

15. A series circuit contains a resistor and an inductor as shown in **FIGURE 1.3.14**. Determine a differential equation for the current $i(t)$ if the resistance is R, the inductance is L, and the impressed voltage is $E(t)$.

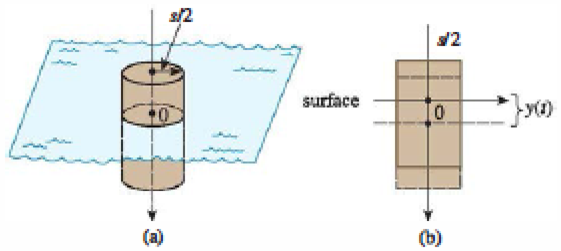


16. A series circuit contains a resistor and a capacitor as shown in **FIGURE 1.3.15**. Determine a differential equation for the charge $q(t)$ on the capacitor if the resistance is R, the capacitance is C, and the impressed voltage is $E(t)$.



≡ Newton's Second Law and Archimedes' Principle

18. A cylindrical barrel s ft in diameter of weight w lb is floating in water as shown in **FIGURE 1.3.17(a)**. After an initial depression, the barrel exhibits an up-and-down bobbing motion along a vertical line. Using Figure 1.3.17(b), determine a differential equation for the vertical displacement $y(t)$ if the origin is taken to be on the vertical axis at the surface of the water when the barrel is at rest. Use **Archimedes' principle**: Buoyancy, or upward force of the water on the barrel, is equal to the weight of the water displaced. Assume that the downward direction is positive, that the weight density of water is 62.4 lb/ft^3, and that there is no resistance between the barrel and the water.



2.7 Linear models (growth/decay, heating/cooling, circuits, mixtures): 3,5,9,15,17,23,25,29,31

≡ Growth and Decay

3. The population of a town grows at a rate proportional to the population present at time t. The initial population of 500 increases by 15% in 10 years. What will the population be in 30 years? How fast is the population growing at $t = 30$?

5. The radioactive isotope of lead, Pb-209, decays at a rate proportional to the amount present at time t and has a half-life of 3.3 hours. If 1 gram of this isotope is present initially, how long will it take for 90% of the lead to decay?

9. When a vertical beam of light passes through a transparent medium, the rate at which its intensity I decreases is proportional to $I(t)$, where t represents the thickness of the medium (in feet). In clear seawater, the intensity 3 feet below the surface is 25% of the initial intensity I_0 of the incident beam. What is the intensity of the beam 15 feet below the surface?

≡ Newton's Law of Cooling/Warming

15. A small metal bar, whose initial temperature was 20°C, is dropped into a large container of boiling water. How long will it take the bar to reach 90°C if it is known that its temperature increased 2° in 1 second? How long will it take the bar to reach 98°C?

17. A thermometer reading 70°F is placed in an oven preheated to a constant temperature. Through a glass window in the oven door, an observer records that the thermometer read 110°F after $\frac{1}{2}$ minute and 145°F after 1 minute. How hot is the oven?

≡ Mixtures

23. A large tank is filled to capacity with 500 gallons of pure water. Brine containing 2 pounds of salt per gallon is pumped into the tank at a rate of 5 gal/min. The well-mixed solution is pumped out at the same rate. Find the number $A(t)$ of pounds of salt in the tank at time t.

25. Solve Problem 23 under the assumption that the solution is pumped out at a faster rate of 10 gal/min. When is the tank empty?

≡ Series Circuits

29. A 30-volt electromotive force is applied to an LR-series circuit in which the inductance is 0.1 henry and the resistance is 50 ohms. Find the current $i(t)$ if $i(0) = 0$. Determine the current as $t \to \infty$.

31. A 100-volt electromotive force is applied to an RC-series circuit in which the resistance is 200 ohms and the capacitance is 10^{-4} farad. Find the charge $q(t)$ on the capacitor if $q(0) = 0$. Find the current $i(t)$.

2.8 Non-linear models (Population dynamics, logistic equation, chemical reaction, leaking tank): 2,3,11,13,17

≡ Logistic Equation

2. The number $N(t)$ of people in a community who are exposed to a particular advertisement is governed by the logistic equation. Initially $N(0) = 500$, and it is observed that $N(1) = 1000$. Solve for $N(t)$ if it is predicted that the limiting number of people in the community who will see the advertisement is 50,000.

3. A model for the population $P(t)$ in a suburb of a large city is given by the initial-value problem

$$\frac{dP}{dt} = P(10^{-1} - 10^{-7}P), \quad P(0) = 5000,$$

where t is measured in months. What is the limiting value of the population? At what time will the population be equal to one-half of this limiting value?

≡ Chemical Reactions

11. Two chemicals A and B are combined to form a chemical C. The rate, or velocity, of the reaction is proportional to the product of the instantaneous amounts of A and B not converted to chemical C. Initially there are 40 grams of A and 50 grams of B, and for each gram of B, 2 grams of A is used. It is observed that 10 grams of C is formed in 5 minutes. How much is formed in 20 minutes? What is the limiting amount of C after a long time? How much of chemicals A and B remains after a long time?

≡ Miscellaneous Nonlinear Models

13. Leaking Cylindrical Tank A tank in the form of a right-circular cylinder standing on end is leaking water through a circular hole in its bottom. As we saw in (10) of Section 1.3, when friction and contraction of water at the hole are ignored, the height h of water in the tank is described by

$$\frac{dh}{dt} = -\frac{A_h}{A_w}\sqrt{2gh},$$

where A_w and A_h are the cross-sectional areas of the water and the hole, respectively.

(a) Solve for $h(t)$ if the initial height of the water is H. By hand, sketch the graph of $h(t)$ and give its interval I of definition in terms of the symbols A_w, A_h, and H. Use $g = 32$ ft/s^2.

(b) Suppose the tank is 10 ft high and has radius 2 ft and the circular hole has radius $\frac{1}{2}$ in. If the tank is initially full, how long will it take to empty?

17. Air Resistance A differential equation governing the velocity v of a falling mass m subjected to air resistance proportional to the square of the instantaneous velocity is

$$m\frac{dv}{dt} = mg - kv^2,$$

where $k > 0$ is the drag coefficient. The positive direction is downward.

(a) Solve this equation subject to the initial condition $v(0) = v_0$.

(b) Use the solution in part (a) to determine the limiting, or terminal, velocity of the mass. We saw how to determine the terminal velocity without solving the DE in Problem 39 in Exercises 2.1.

(c) If distance s, measured from the point where the mass was released above ground, is related to velocity v by $ds/dt = v(t)$, find an explicit expression for $s(t)$ if $s(0) = 0$.

17.1 Complex numbers: 1,3,7,11,15,25,27,29,31,35,39

In Problems 1–26, write the given number in the form $a + ib$.

1. $2i^3 - 3i^2 + 5i$

3. i^8

7. $i(5 + 7i)$

11. $(2 + 3i)^2$

15. $\dfrac{2 - 4i}{3 + 5i}$

25. $\left(\dfrac{i}{3 - i}\right)\left(\dfrac{1}{2 + 3i}\right)$

In Problems 27–32, let $z = x + iy$. Find the indicated expression.

27. $\text{Re}(1/z)$

29. $\text{Im}(2z + 4\bar{z} - 4i)$

31. $|z - 1 - 3i|$

In Problems 33–38, use Definition 17.1.2 to find a complex number z satisfying the given equation.

35. $z^2 = i$

In Problems 39 and 40, determine which complex number is closer to the origin.

39. $10 + 8i, \quad 11 - 6i$

17.2 Powers and Roots: 3,7,9,15,21,31,33,35

Write the given complex number in polar form.

3. $-3i$

7. $-\sqrt{3}+i$

9. $\dfrac{3}{-1+i}$

Find $z_1 z_2$ *and* z_1/z_2. Write the number in the form $a + ib$.

15. $z_1 = 2\left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right)$,

$z_2 = 4\left(\cos\frac{3\pi}{8} + i\sin\frac{3\pi}{8}\right)$

In Problems 21–26, use (8) to compute the indicated power.

21. $(1+\sqrt{3}i)^9$

In Problems 27–32, use (10) to compute all roots. Sketch these roots on an appropriate circle centered at the origin.

31. $(-1+\sqrt{3}i)^{1/2}$

In Problems 33 and 34, find all solutions of the given equation.

33. $z^4 + 1 = 0$

In Problems 35 and 36, express the given complex number first in polar form and then in the form $a + ib$.

35. $\left(\cos\frac{\pi}{9} + i\sin\frac{\pi}{9}\right)^{12}\left[2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\right]^5$

Answers:

1.1 Definition and Terminology

1. Second order; linear
3. Fourth order; linear
5. Second order; nonlinear because of $\left(\frac{dy}{dx}\right)^2$ or $\sqrt{1+\left(\frac{dy}{dx}\right)^2}$
6. Second order; nonlinear because of R^2
8. Second order; nonlinear because of $\dot{x}^2$
10. Writing the differential equation in the form $x(dy/dx) + y^2 = 1$, we see that it is nonlinear in y because of y^2. However, writing it in the form $(y^2 - 1)(dx/dy) + x = 0$, we see that it is linear in x.
11. From $y = e^{-\frac{x}{2}}$ we obtain $y' = -\frac{1}{2}e^{-\frac{x}{2}}$. Then $2y' + y = -e^{-\frac{x}{2}} + e^{-\frac{x}{2}} = 0$.
13. From $y = e^{3x}\cos 2x$ we obtain $y' = 3e^{3x}\cos 2x - 2e^{3x}\sin 2x$ and $y'' = 5e^{3x}\cos 2x - 12e^{3x}\sin 2x$, so that $y'' = 6y' + 13y = 0$.
14. From $y = -\cos x \ln(\sec x + \tan x)$ we obtain $y' = -1 + \sin x \ln(\sec x + \tan x)$ and $y'' = \tan x + \cos x \ln(\sec x + \tan x)$. Then $y'' + y = \tan x$.

21. Differentiating $P = c_1e^t/(1 + c_1e^t)$ we obtain

$$\frac{dP}{dt} = \frac{(1+c_1e^t)\,c_1e^t - c_1e^t \cdot c_1e^t}{(1+c_1e^t)^2} = \frac{c_1e^t}{1+c_1e^t}\,\frac{[(1+c_1e^t) - c_1e^t]}{1+c_1e^t}$$

$$= \frac{c_1e^t}{1+c_1e^t}\left[1 - \frac{c_1e^t}{1+c_1e^t}\right] = P(1-P).$$

23. From $y = c_1e^{2x} + c_2xe^{2x}$ we obtain $\frac{uy}{dx} = (2c_1 + c_2)e^{2x} + 2c_2xe^{2x}$ and $\frac{u\ y}{dx^2} = (4c_1 + 4c_2)e^{2x} + 4c_2xe^{2x}$, so that

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = (4c_1 + 4c_2 - 8c_1 - 4c_2 + 4c_1)e^{2x} + (4c_2 - 8c_2 + 4c_2)xe^{2x} = 0.$$

1.2 Initial Value Problems

7. From the initial conditions, we obtain the system:$c_1 = -1, c_2 = 8$. The solution is $x = -\cos t + 8\sin t$.

9. From the initial conditions we obtain

$$\frac{\sqrt{3}}{2}c_1 + \frac{1}{2}c_2 = \frac{1}{2}$$

$$-\frac{1}{2}c_1 + \frac{\sqrt{3}}{2}c_2 = 0.$$

Solving, we find $c_1 = \sqrt{3}/4$ and $c_2 = 1/4$. The solution of the initial-value problem is

$$x = (\sqrt{3}/4)\cos t + (1/4)\sin t.$$

11. From the initial conditions we obtain

$$c_1 + c_2 = 1$$

$$c_1 - c_2 = 2.$$

Solving, we find $c_1 = \frac{3}{2}$ and $c_2 = -\frac{1}{2}$. The solution of the initial-value problem is $y = \frac{3}{2}e^x - \frac{1}{2}e^{-x}$.

12. From the initial conditions we obtain

$$ec_1 + e^{-1}c_2 = 0$$
$$ec_1 - e^{-1}c_2 = e.$$

Solving, we find $c_1 = \frac{1}{2}$ and $c_2 = -\frac{1}{2}e^2$. The solution of the initial-value problem is

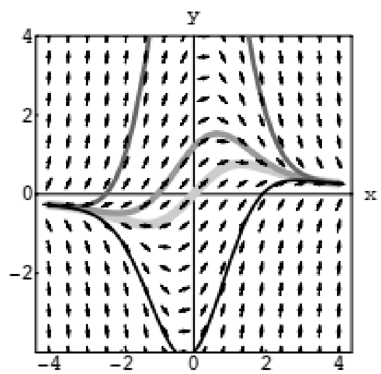
$$y = \frac{1}{2}e^x - \frac{1}{2}e^2e^{-x} = \frac{1}{2}e^x - \frac{1}{2}e^{2-x}.$$

17. For $f(x,y) = y^{2/3}$ we have $\dfrac{\partial f}{\partial y} = \dfrac{2}{3}y^{-1/3}$. Thus, the differential equation will have a unique solution in any rectangular region of the plane where $y \neq 0$.

18. For $f(x,y) = \sqrt{xy}$ we have $\partial f/\partial y = \frac{1}{2}\sqrt{x/y}$. Thus, the differential equation will have a unique solution in any region where $x > 0$ and $y > 0$ or where $x < 0$ and $y < 0$.

2.1 Solution curves without a solution

3.



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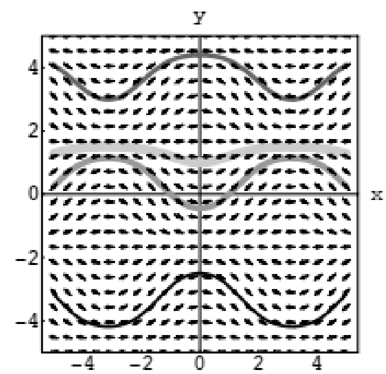
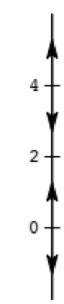
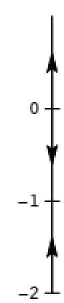


26. Solving $y(2-y)(4-y) = 0$ we obtain the critical points 0, 2, and 4. From the phase portrait we see that 2 is asymptotically stable (attractor) and 0 and 4 are unstable (repellers).



27. Solving $y\ln(y+2) = 0$ we obtain the critical points -1 and 0. From the phase portrait we see that -1 is asymptotically stable (attractor) and 0 is unstable (repeller).



2.2 Separable Equations

7. From $e^{-2y}dy = e^{3x}dx$ we obtain $3e^{-2y} + 2e^{3x} = c$.

9. From $\left(y + 2 + \frac{1}{y}\right) dy = x^2 \ln x \, dx$ we obtain $\frac{y^2}{2} + 2y + \ln|y| = \frac{x^3}{3}\ln|x| - \frac{1}{9}x^3 + c$.

13. From $\frac{e^y}{(e^y+1)^2}\,dy = \frac{-e^x}{(e^x+1)^3}\,dx$ we obtain $-(e^y+1)^{-1} = \frac{1}{2}(e^x+1)^{-2} + c$.

19. From $\frac{y-2}{y+3}\,dy = \frac{x-1}{x+4}\,dx$ or $\left(1 - \frac{5}{y+3}\right) dy = \left(1 - \frac{5}{x+4}\right) dx$ we obtain $y - 5\ln|y+3| = x - 5\ln|x+4| + c$ or $\left(\frac{x+4}{y+3}\right)^5 = c_1 e^{x-y}$.

25. From $\frac{1}{y}\,dy = \frac{1-x}{x^2}\,dx = \left(\frac{1}{x^2} - \frac{1}{x}\right) dx$ we obtain $\ln|y| = -\frac{1}{x} - \ln|x| = c$ or $xy = c_1 e^{-1/x}$. Using $y(-1) = -1$ we find $c_1 = e^{-1}$. The solution of the initial-value problem is $xy = e^{-1-1/x}$ or $y = e^{-(1+1/x)}/x$.

27. Separating variables and integrating we obtain

$$\frac{dx}{\sqrt{1-x^2}} - \frac{dy}{\sqrt{1-y^2}} = 0 \quad \text{and} \quad \sin^{-1} x - \sin^{-1} y = c.$$

Setting $x = 0$ and $y = \sqrt{3}/2$ we obtain $c = -\pi/3$. Thus, an implicit solution of the initial-value problem is $\sin^{-1} x - \sin^{-1} y = \pi/3$. Solving for y and using an addition formula from trigonometry, we get

$$y = \sin\left(\sin^{-1} x + \frac{\pi}{3}\right) = x\cos\frac{\pi}{3} + \sqrt{1-x^2}\sin\frac{\pi}{3} = \frac{x}{2} + \frac{\sqrt{3}\sqrt{1-x^2}}{2}.$$

2.3 Linear Equations

7. For $y' + \frac{1}{x}y = \frac{1}{x^2}$ an integrating factor is $e^{\int (1/x)dx} = x$ so that $\frac{d}{dx}[xy] = \frac{1}{x}$ and $y = \frac{1}{x}\ln x + \frac{c}{x}$ for $0 < x < \infty$.

9. For $y' - \frac{1}{x}y = x\sin x$ an integrating factor is $e^{-\int (1/x)dx} = \frac{1}{x}$ so that $\frac{d}{dx}\left[\frac{1}{x}y\right] = \sin x$ and $y = cx - x\cos x$ for $0 < x < \infty$.

23. For $y' + \left(3 + \frac{1}{x}\right) y = \frac{e^{-3x}}{x}$ an integrating factor is $e^{\int [3+(1/x)]dx} = xe^{3x}$ so that $\frac{d}{dx}\left[xe^{3x}y\right] = 1$ and $y = e^{-3x} + \frac{ce^{-3x}}{x}$ for $0 < x < \infty$. The transient term is ce^{-3x}/x.

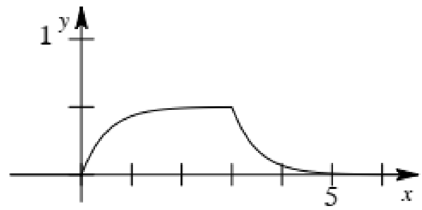
27. For $\dfrac{di}{dt}+\dfrac{R}{L}i=\dfrac{E}{L}$ an integrating factor is $e^{\int (R/L)\,dt}=e^{Rt/L}$ so that $\dfrac{d}{dt}\left[e^{Rt/L}i\right]=\dfrac{E}{L}e^{Rt/L}$ and $i=\dfrac{E}{R}+ce^{-Rt/L}$ for $-\infty<t<\infty$. If $i(0)=i_0$ then $c=i_0-E/R$ and $i=\dfrac{E}{R}+\left(i_0-\dfrac{E}{R}\right)e^{-Rt/L}$.

31. For $y'+2y=f(x)$ an integrating factor is e^{2x} so that

$$ye^{2x}=\begin{cases}\frac{1}{2}e^{2x}+c_1, & 0\le x\le 3\\ c_2, & x>3.\end{cases}$$

If $y(0)=0$ then $c_1=-1/2$ and for continuity we must have $c_2=\frac{1}{2}e^6-\frac{1}{2}$ so that

$$y=\begin{cases}\frac{1}{2}(1-e^{-2x}), & 0\le x\le 3\\ \frac{1}{2}(e^6-1)e^{-2x}, & x>3.\end{cases}$$



2.4 Exact Equations, integrating factors

3. Let $M=5x+4y$ and $N=4x-8y^3$ so that $M_y=4=N_x$. From $f_x=5x+4y$ we obtain $f=\frac{5}{2}x^2+4xy+h(y)$, $h'(y)=-8y^3$, and $h(y)=-2y^4$. A solution is $\frac{5}{2}x^2+4xy-2y^4=c$.

5. Let $M=2y^2x-3$ and $N=2yx^2+4$ so that $M_y=4xy=N_x$. From $f_x=2y^2x-3$ we obtain $f=x^2y^2-3x+h(y)$, $h'(y)=4$, and $h(y)=4y$. A solution is $x^2y^2-3x+4y=c$.

9. Let $M=y^3-y^2\sin x-x$ and $N=3xy^2+2y\cos x$ so that $M_y=3y^2-2y\sin x=N_x$. From $f_x=y^3-y^2\sin x-x$ we obtain $f=xy^3+y^2\cos x-\frac{1}{2}x^2+h(y)$, $h'(y)=0$, and $h(y)=0$. A solution is $xy^3+y^2\cos x-\frac{1}{2}x^2=c$.

15. Let $M=x^2y^3-1/\left(1+9x^2\right)$ and $N=x^3y^2$ so that $M_y=3x^2y^2=N_x$. From $f_x=x^2y^3-1/\left(1+9x^2\right)$ we obtain $f=\frac{1}{3}x^3y^3-\frac{1}{3}\arctan(3x)+h(y)$, $h'(y)=0$, and $h(y)=0$. A solution is $x^3y^3-\arctan(3x)=c$.

27. Equating $M_y=3y^2+4kxy^3$ and $N_x=3y^2+40xy^3$ we obtain $k=10$.

29. Let $M=-x^2y^2\sin x+2xy^2\cos x$ and $N=2x^2y\cos x$ so that $M_y=-2x^2y\sin x+4xy\cos x=N_x$. From $f_y=2x^2y\cos x$ we obtain $f=x^2y^2\cos x+h(y)$, $h'(y)=0$, and $h(y)=0$. A solution of the differential equation is $x^2y^2\cos x=c$.

31. We note that $(M_y-N_x)/N=1/x$, so an integrating factor is $e^{\int dx/x}=x$. Let $M=2xy^2+3x^2$ and $N=2x^2y$ so that $M_y=4xy=N_x$. From $f_x=2xy^2+3x^2$ we obtain $f=x^2y^2+x^3+h(y)$, $h'(y)=0$, and $h(y)=0$. A solution of the differential equation is $x^2y^2+x^3=c$.

2.5 Solutions by Substitution (Bernoulli, homogeneous, linear substitution)

5. Letting $y = ux$ we have

$$\begin{aligned}
(u^2x^2 + ux^2)\,dx - x^2(u\,dx + x\,du) &= 0 \\
u^2\,dx - x\,du &= 0 \\
\frac{dx}{x} - \frac{du}{u^2} &= 0 \\
\ln|x| + \frac{1}{u} &= c \\
\ln|x| + \frac{x}{y} &= c \\
y\ln|x| + x &= cy.
\end{aligned}$$

7. Letting $y = ux$ we have

$$\begin{aligned}
(ux - x)\,dx - (ux + x)(u\,dx + x\,du) &= 0 \\
(u^2 + 1)\,dx + x(u + 1)\,du &= 0 \\
\frac{dx}{x} + \frac{u + 1}{u^2 + 1}\,du &= 0 \\
\ln|x| + \frac{1}{2}\ln\left(u^2 + 1\right) + \tan^{-1} u &= c \\
\ln x^2\left(\frac{y^2}{x^2} + 1\right) + 2\tan^{-1}\frac{y}{x} &= c_1 \\
\ln\left(x^2 + y^2\right) + 2\tan^{-1}\frac{y}{x} &= c_1.
\end{aligned}$$

9. Letting $y = ux$ we have

$$\begin{aligned}
-ux\,dx + (x + \sqrt{u}\,x)(u\,dx + x\,du) &= 0 \\
(x^2 + x^2\sqrt{u}\,)\,du + xu^{3/2}\,dx &= 0 \\
\left(u^{-3/2} + \frac{1}{u}\right)du + \frac{dx}{x} &= 0 \\
-2u^{-1/2} + \ln|u| + \ln|x| &= c \\
\ln|y/x| + \ln|x| &= 2\sqrt{x/y} + c \\
y(\ln|y| - c)^2 &= 4x.
\end{aligned}$$

13. Letting $y = ux$ we have

$$\begin{aligned}
(x + uxe^u)\,dx - xe^u(u\,dx + x\,du) &= 0 \\
dx - xe^u\,du &= 0 \\
\frac{dx}{x} - e^u\,du &= 0 \\
\ln|x| - e^u &= c \\
\ln|x| - e^{y/x} &= c.
\end{aligned}$$

Using $y(1) = 0$ we find $c = -1$. The solution of the initial-value problem is $\ln|x| = e^{y/x} - 1$.

17. From $y' + y = xy^4$ and $w = y^{-3}$ we obtain $\frac{aw}{dx} - 3w = -3x$. An integrating factor is e^{-3x} so that $e^{-3x}w = xe^{-3x} + \frac{1}{3}e^{-3x} + c$ or $y^{-3} = x + \frac{1}{3} + ce^{3x}$.

19. From $y' - \frac{1}{t}y = -\frac{1}{t^2}y^2$ and $w = y^{-1}$ we obtain $\frac{dw}{dt} + \frac{1}{t}w = \frac{1}{t^2}$. An integrating factor is t so that $tw = \ln t + c$ or $y^{-1} = \frac{1}{t}\ln t + \frac{c}{t}$. Writing this in the form $\frac{t}{y} = \ln t + c$, we see that the solution can also be expressed in the form $e^{t/y} = c_1 t$.

21. From $y' - \frac{2}{x}y = \frac{3}{x^2}y^4$ and $w = y^{-3}$ we obtain $\frac{dw}{dx} + \frac{6}{x}w = -\frac{9}{x^2}$. An integrating factor is x^6 so that

$x^6 w = -\frac{9}{5}x^5 + c$ or $y^{-3} = -\frac{9}{5}x^{-1} + cx^{-6}$. If $y(1) = \frac{1}{2}$ then $c = \frac{49}{5}$ and $y^{-3} = -\frac{9}{5}x^{-1} + \frac{49}{5}x^{-6}$.

25. Let $u = x + y$ so that $du/dx = 1 + dy/dx$. Then $\frac{du}{dx} - 1 = \tan^2 u$ or $\cos^2 u\, du = dx$. Thus $\frac{1}{2}u + \frac{1}{4}\sin 2u = x + c$

or $2u + \sin 2u = 4x + c_1$, and $2(x + y) + \sin 2(x + y) = 4x + c_1$ or $2y + \sin 2(x + y) = 2x + c_1$.

27. Let $u = y - 2x + 3$ so that $du/dx = dy/dx - 2$. Then $\frac{du}{dx} + 2 = 2 + \sqrt{u}$ or $\frac{1}{\sqrt{u}}\,du = dx$. Thus $2\sqrt{u} = x + c$ and $2\sqrt{y - 2x + 3} = x + c$.

1.3 Differential Equations as Mathematical Models

1. $\frac{dP}{dt} = kP + r; \qquad \frac{dP}{dt} = kP - r$

2. Let b be the rate of births and d the rate of deaths. Then $b = k_1 P$ and $d = k_2 P$. Since $dP/dt = b - d$, the differential equation is $dP/dt = k_1 P - k_2 P$.

3. Let b be the rate of births and d the rate of deaths. Then $b = k_1 P$ and $d = k_2 P^2$. Since $dP/dt = b - d$, the differential equation is $dP/dt = k_1 P - k_2 P^2$.

5. From the graph in the text we estimate $T_0 = 180°$ and $T_m = 75°$. We observe that when $T = 85$, $dT/dt \approx -1$. From the differential equation we then have

$$k = \frac{dT/dt}{T - T_m} = \frac{-1}{85 - 75} = -0.1.$$

7. The number of students with the flu is x and the number not infected is $1000 - x$, so $dx/dt = kx(1000 - x)$.

9. The rate at which salt is leaving the tank is

$$R_{out}\ (3\text{ gal/min}) \cdot \left(\frac{A}{300}\text{ lb/gal}\right) = \frac{A}{100}\text{ lb/min.}$$

Thus $dA/dt = A/100$. The initial amount is $A(0) = 50$.

10. The rate at which salt is entering the tank is

$$R_{in} = (3\text{ gal/min}) \cdot (2\text{ lb/gal}) = 6\text{ lb/min.}$$

Since the solution is pumped out at a slower rate, it is accumulating at the rate of $(3-2)\text{gal/min} = 1$ gal/min. After t minutes there are $300+t$ gallons of brine in the tank. The rate at which salt is leaving is

$$R_{out} = (2 \text{ gal/min}) \cdot \left(\frac{A}{300+t} \text{ lb/gal} \right) = \frac{2A}{300+t} \text{ lb/min.}$$

The differential equation is

$$\frac{dA}{dt} = 6 - \frac{2A}{300+t}.$$

13. The volume of water in the tank at time t is $V = A_w h$. The differential equation is then

$$\frac{dh}{dt} = \frac{1}{A_w}\frac{dV}{dt} = \frac{1}{A_w}\left(-cA_h\sqrt{2gh}\right) = -\frac{cA_h}{A_w}\sqrt{2gh}.$$

Using $A_h = \pi \left(\frac{2}{12}\right)^2 = \frac{\pi}{36}$, $A_w = 10^2 = 100$, and $g = 32$, this becomes

$$\frac{dh}{dt} = -\frac{c\pi/36}{100}\sqrt{64h} = -\frac{c\pi}{450}\sqrt{h}.$$

15. Since $i = dq/dt$ and $L\,d^2q/dt^2 + R\,dq/dt = E(t)$, we obtain $L\,di/dt + Ri = E(t)$.

16. By Kirchhoff's second law we obtain $R\dfrac{dq}{dt} + \dfrac{1}{C}q = E(t)$.

19. The net force acting on the mass is

$$F = ma = m\,\frac{d^2x}{dt^2} = -k(s+x) + mg = -kx + mg - ks.$$

Since the condition of equilibrium is $mg = ks$, the differential equation is

$$m\,\frac{d^2x}{dt^2} = -kx.$$

2.7 Linear models (growth/decay, heating/cooling, circuits, mixtures)

3. Let $P = P(t)$ be the population at time t. Then $dP/dt = kP$ and $P = ce^{kt}$. From $P(0) = c = 500$ we see that $P = 500e^{kt}$. Since 15% of 500 is 75, we have $P(10) = 500e^{10k} = 575$. Solving for k, we get $k = \frac{1}{10}\ln\frac{575}{500} = \frac{1}{10}\ln 1.15$. When $t = 30$,

$$P(30) = 500e^{(1/10)(\ln 1.15)30} = 500e^{3\ln 1.15} = 760 \text{ years}$$

and

$$P'(30) = kP(30) = \frac{1}{10}(\ln 1.15)760 = 10.62 \text{ persons/year.}$$

5. Let $A = A(t)$ be the amount of lead present at time t. From $dA/dt = kA$ and $A(0) = 1$ we obtain $A = e^{kt}$. Using $A(3.3) = 1/2$ we find $k = \frac{1}{3.3}\ln(1/2)$. When 90% of the lead has decayed, 0.1 grams will remain. Setting $A(t) = 0.1$ we have $e^{t(1/3.3)\ln(1/2)} = 0.1$, so

$$\frac{t}{3.3}\ln\frac{1}{2} = \ln 0.1 \qquad \text{and} \qquad t = \frac{3.3\ln 0.1}{\ln(1/2)} \approx 10.96 \text{ hours.}$$

9. Let $I = I(t)$ be the intensity, t the thickness, and $I(0) = I_0$. If $dI/dt = kI$ and $I(3) = 0.25I_0$, then $I = I_0e^{kt}$, $k = \frac{1}{3}\ln 0.25$, and $I(15) = 0.00098I_0$.

15. Assume that $dT/dt = k(T - 100)$ so that $T = 100 + ce^{kt}$. If $T(0) = 20°$ and $T(1) = 22°$, then $c = -80$ and $k = \ln(39/40)$ so that $T(t) = 90°$, which implies $t = 82.1$ seconds. If $T(t) = 98°$ then $t = 145.7$ seconds.

17. Using separation of variables to solve $dT/dt = k(T - T_m)$ we get $T(t) = T_m + ce^{kt}$. Using $T(0) = 70$ we find $c = 70 - T_m$, so $T(t) = T_m + (70 - T_m)e^{kt}$. Using the given observations, we obtain

$$T\left(\frac{1}{2}\right) = T_m + (70 - T_m)e^{k/2} = 110$$
$$T(1) = T_m + (70 - T_m)e^{k} = 145.$$

Then, from the first equation, $e^{k/2} = (110 - T_m)/(70 - T_m)$ and

$$e^k = (e^{k/2})^2 = \left(\frac{110 - T_m}{70 - T_m}\right)^2 = \frac{145 - T_m}{70 - T_m}$$
$$\frac{(110 - T_m)^2}{70 - T_m} = 145 - T_m$$
$$12100 - 220T_m + T_m^2 = 10150 - 250T_m + T_m^2$$
$$T_m = 390.$$

The temperature in the oven is 390°.

23. From
$$\frac{dA}{dt} = 10 - \frac{10A}{500 - (10 - 5)t} = 10 - \frac{2A}{100 - t}$$
we obtain $A = 1000 - 10t + c(100 - t)^2$. If $A(0) = 0$ then $c = -\frac{1}{10}$. The tank is empty in 100 minutes.

25. From
$$\frac{dA}{dt} = 3 - \frac{4A}{100 + (6 - 4)t} = 3 - \frac{2A}{50 + t}$$
we obtain $A = 50 + t + c(50 + t)^{-2}$. If $A(0) = 10$ then $c = -100{,}000$ and $A(30) = 64.38$ pounds.

29. Assume $R\,dq/dt + (1/C)q = E(t)$, $R = 200$, $C = 10^{-4}$, and $E(t) = 100$ so that $q = 1/100 + ce^{-50t}$. If $q(0) = 0$ then $c = -1/100$ and $i = \frac{1}{2}e^{-50t}$.

31. For $0 \le t \le 20$ the differential equation is $20\,di/dt + 2i = 120$. An integrating factor is $e^{t/10}$, so $(d/dt)[e^{t/10}i] = 6e^{t/10}$ and $i = 60 + c_1e^{-t/10}$. If $i(0) = 0$ then $c_1 = -60$ and $i = 60 - 60e^{-t/10}$. For $t > 20$ the differential equation is $20\,di/dt + 2i = 0$ and $i = c_2e^{-t/10}$. At $t = 20$ we want $c_2e^{-2} = 60 - 60e^{-2}$ so that $c_2 = 60\left(e^2 - 1\right)$. Thus
$$i(t) = \begin{cases} 60 - 60e^{-t/10}, & 0 \le t \le 20 \\ 60\left(e^2 - 1\right)e^{-t/10}, & t > 20. \end{cases}$$

2.8 Non-linear models (Population dynamics, logistic equation, chemical reaction, leaking tank

2. From $dN/dt = N(a - bN)$ and $N(0) = 500$ we obtain

$$N = \frac{500a}{500b + (a - 500b)e^{-at}}.$$

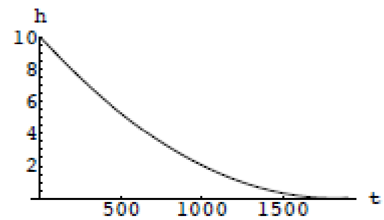
Since $\lim_{t\to\infty} N = a/b = 50{,}000$ and $N(1) = 1000$ we have $a = 0.7033$, $b = 0.00014$, and $N = 50{,}000/(1 + 99e^{-0.7033t})$.

3. From $dP/dt = P\left(10^{-1} - 10^{-7}P\right)$ and $P(0) = 5000$ we obtain $P = 500/(0.0005 + 0.0995e^{-0.1t})$ so that $P \to 1{,}000{,}000$ as $t \to \infty$. If $P(t) = 500{,}000$ then $t = 52.9$ months.

11. (a) The initial-value problem is $dh/dt = -8A_h\sqrt{h}\,/A_w$, $h(0) = H$. Separating variables and integrating we have

$$\frac{dh}{\sqrt{h}} = -\frac{8A_h}{A_w}\,dt \quad \text{and} \quad 2\sqrt{h} = -\frac{8A_h}{A_w}t + c.$$

Using $h(0) = H$ we find $c = 2\sqrt{H}$, so the solution of the initial-value problem is $\sqrt{h(t)} = (A_w\sqrt{H} - 4A_h t)/A_w$, where $A_w\sqrt{H} - 4A_h t \ge 0$. Thus,



$$h(t) = (A_w\sqrt{H} - 4A_h t)^2/A_w^2 \quad \text{for} \quad 0 \le t \le A_w H/4A_h.$$

(b) Identifying $H = 10$, $A_w = 4\pi$, and $A_h = \pi/576$ we have $h(t) = t^2/331{,}776 - (\sqrt{5/2}\,/144)t + 10$. Solving $h(t) = 0$ we see that the tank empties in $576\sqrt{10}$ seconds or 30.36 minutes.

13. (a) Separating variables and integrating gives

$$6h^{3/2}dh = -5t \quad \text{and} \quad \frac{12}{5}h^{5/2} = -5t + c.$$

Using $h(0) = 20$ we find $c = 1920\sqrt{5}$, so the solution of the initial-value problem is $h(t) = \left(800\sqrt{5} - \frac{25}{12}t\right)^{2/5}$. Solving $h(t) = 0$ we see that the tank empties in $384\sqrt{5}$ seconds or 14.31 minutes.

(b) When the height of the water is h, the radius of the top of the water is $r = h\tan 30° = h/\sqrt{3}$ and $A_w = \pi h^2/3$. The differential equation is

$$\frac{dh}{dt} = -c\frac{A_h}{A_w}\sqrt{2gh} = -0.6\frac{\pi(2/12)^2}{\pi h^2/3}\sqrt{64h} = -\frac{2}{5h^{3/2}}.$$

Separating variables and integrating gives

$$5h^{3/2}dh = -2\,dt \quad \text{and} \quad 2h^{5/2} = -2t + c.$$

Using $h(0) = 9$ we find $c = 486$, so the solution of the initial-value problem is $h(t) = (243 - t)^{2/5}$. Solving $h(t) = 0$ we see that the tank empties in 24.3 seconds or 4.05 minutes.

17. (a) Let ρ be the weight density of the water and V the volume of the object. Archimedes' principle states that the upward buoyant force has magnitude equal to the weight of the water displaced. Taking the positive direction to be down, the differential equation is

$$m\frac{dv}{dt} = mg - kv^2 - \rho V.$$

(b) Using separation of variables we have

$$\frac{m\,dv}{(mg-\rho V) - kv^2} = dt$$

$$\frac{m}{\sqrt{k}}\frac{\sqrt{k}\,dv}{(\sqrt{mg-\rho V}\,)^2 - (\sqrt{k}\,v)^2} = dt$$

$$\frac{m}{\sqrt{k}}\frac{1}{\sqrt{mg-\rho V}}\tanh^{-1}\frac{\sqrt{k}\,v}{\sqrt{mg-\rho V}} = t + c.$$

Thus

$$v(t) = \sqrt{\frac{mg-\rho V}{k}}\tanh\left(\frac{\sqrt{kmg - k\rho V}}{m}t + c_1\right).$$

(c) Since $\tanh t \to 1$ as $t \to \infty$, the terminal velocity is $\sqrt{(mg-\rho V)/k}$.

17.1 Complex numbers

1. $3 + 3i$

3. $i^8 = (i^2)^4 = (-1)^4 = 1$

7. $-7 + 5i$

11. $-5 + 12i$

15. $\frac{2-4i}{3+5i}\cdot\frac{3-5i}{3-5i} = \frac{-14-22i}{34} = -\frac{7}{17} - \frac{11}{17}i$

25. $\frac{i}{9+7i}\cdot\frac{9-7i}{9-7i} = \frac{7+9i}{130} = \frac{7}{130} + \frac{9}{130}i$

27. $\frac{x}{x^2+y^2}$

29. $-2y - 4$

31. $\sqrt{(x-1)^2 + (y-3)^2}$

35. $x^2 - y^2 + 2xyi = 0 + i$ implies $x^2 - y^2 = 0$ and $2xy = 1$. Now $y = x$ implies $2x^2 = 1$ and so $x = \pm 1/\sqrt{2}$. The choice $y = -x$ gives $-2x^2 = 1$ which has no real solution. Hence $z = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$ and $z = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$.

39. $|z_1 - z_2| = |(x_1 - x_2) + i(y_1 - y_2)| = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$ which is the distance formula in the plane.

17.2 Powers and Roots

3. $3\left(\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}\right)$

7. $2\left(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right)$

9. $\frac{3\sqrt{2}}{2}\left(\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}\right)$

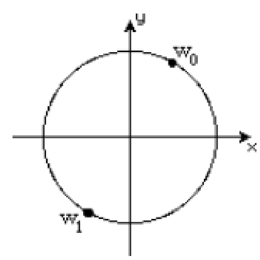
15. $z_1z_2=8\left[\cos\left(\frac{\pi}{8}+\frac{3\pi}{8}\right)+i\sin\left(\frac{\pi}{8}+\frac{3\pi}{8}\right)\right]=8i;$

$$\frac{z_1}{z_2}=\frac{1}{2}\left[\cos\left(\frac{\pi}{8}-\frac{3\pi}{8}\right)+i\sin\left(\frac{\pi}{8}-\frac{3\pi}{8}\right)\right]=\frac{\sqrt{2}}{4}-\frac{\sqrt{2}}{4}i$$

21. $2^9\left[\cos\frac{9\pi}{3}+i\sin\frac{9\pi}{3}\right]=-512$

31. $(-1+\sqrt{3}i)^{1/2}=2^{1/2}\left[\cos\left(\frac{\pi}{3}+k\pi\right)+i\sin\left(\frac{\pi}{3}+k\pi\right)\right],\quad k=0,\,1$

$$w_0=2^{1/2}\left[\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right]=\frac{\sqrt{2}}{2}+\frac{\sqrt{6}}{2}i$$
$$w_2=2^{1/2}\left[\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}\right]=-\frac{\sqrt{2}}{2}-\frac{\sqrt{6}}{2}i$$



33. The solutions are the four fourth roots of -1;

$$w_k=\cos\frac{\pi+2k\pi}{4}+i\sin\frac{\pi+2k\pi}{4},\quad k=0,\,1,\,2,\,3.$$

We have

$$w_1=\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}=\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i \qquad w_3=\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}=-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i$$
$$w_2=\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}=-\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i \qquad w_4=\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}=\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i.$$

35. $\left(\cos\frac{\pi}{9}+i\sin\frac{\pi}{9}\right)^{12}\left[2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)\right]^5=2^5\left(\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}\right)\left(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right)$

$$=32\left[\cos\left(\frac{4\pi}{3}+\frac{5\pi}{6}\right)+i\sin\left(\frac{4\pi}{3}+\frac{5\pi}{6}\right)\right]$$
$$=32\left(\cos\frac{13\pi}{6}+i\sin\frac{13\pi}{6}\right)=32\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)=16\sqrt{3}+16i$$