6 marks

Solution for main final

Q1: Solve for x : $4^x - 2^{x+1} = 24$.

Solution:

$$4^x - 2^{x+1} = 24$$

$$\Rightarrow (2^x)^2 - 2\cdot 2^x - 24 = 0$$

let $2^x = t > 0$

$$\Rightarrow t^2 - 2t - 24 = 0$$

$$(t+4)(t-6) = 0$$

$t = -4$ (reject)

$t = 6$

$2^x = 6 \Rightarrow x = \log_2 6$

8 marks

Q2: Given the function $f = 2 - \ln(1+e^{2x})$

1). find the inverse function $f^{-1}(x)$;

2). determine the domain and range of $f(x)$ and of $f^{-1}(x)$

Solution:

1). $$y = 2 - \ln(1+e^{2x})$$

$$\ln(1+e^{2x}) = 2-y$$

$$1+e^{2x} = e^{2-y}$$

$$e^{2x} = e^{2-y} - 1$$

$$2x = \ln(e^{2-y} - 1)$$

$$x = \frac{1}{2}\ln(e^{2-y} - 1)$$

so $$f^{-1}(x) = \frac{1}{2}\ln(e^{2-x} - 1)$$

2). $f = 2 - \ln(1+e^{2x})$: domain ?

$1+e^{2x} > 0 \Rightarrow x \in (-\infty, +\infty)$

$f^{-1} = \frac{1}{2}\ln(e^{2-x} - 1)$: domain ?

$e^{2-x} - 1 > 0 \Rightarrow e^{2-x} > 1 = e^0$

$\Rightarrow 2 - x > 0$

$\Rightarrow x < 2$

Thus,

f: domain : $x \in (-\infty, \infty)$
range $y \in (-\infty, 2)$

f^{-1}: domain : $x \in (-\infty, 2)$
range : $y \in (-\infty, \infty)$

8 marks

Q3: Find all horizontal and all vertical asymptotes of the graph.

$$f(x) = \frac{\sqrt{16x^2+1}}{x^2-16} \cdot \frac{x^2-4x}{x+4}$$

Solution:

$$f(x) = \frac{\sqrt{16x^2+1}}{\cancel{(x-4)}(x+4)} \cdot \frac{x\cancel{(x-4)}}{(x+4)}$$

$$= \frac{x\sqrt{16x^2+1}}{(x+4)^2}$$

$$= \frac{x\sqrt{16x^2+1}}{x^2+8x+16}$$

1). H.A.

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{x\sqrt{16x^2+1}}{x^2+8x+16}$$ divide by x^2

$$= \lim_{x\to+\infty} \frac{1\cdot\sqrt{16+\frac{1}{x}}}{1+\frac{8}{x}+\frac{16}{x^2}} = \frac{\sqrt{16}}{1} = 4.$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x\sqrt{16x^2+1}}{x^2+8x+16}$$

$$= \lim_{x\to-\infty} \frac{\boxed{-}\sqrt{16+\frac{1}{x}}}{1+\frac{8}{x}+\frac{16}{x^2}} = \frac{-\sqrt{16}}{1} = -4$$

H.A. : $y = 4$ and $y = -4$

2) V.A. $(x^2-16)(x+4) = 0$

$\Rightarrow$ $x = 4$ or $x = -4$

V.A. : $x = 4$ and $x = -4$

Q4: Find the limit. 6 marks

if the limit does not exist, explain why.

$$\lim_{x \to -1} \frac{\sqrt{x^2-x+2}+x-1}{x^2-1}$$

Solution:

$$\frac{\sqrt{x^2-x+2}+x-1}{x^2-1} \cdot \frac{\sqrt{x^2-x+2}-(x-1)}{\sqrt{x^2-x+2}-(x-1)}$$

$$= \frac{\cancel{x+1}}{(\cancel{x+1})(x-1)} \cdot \frac{1}{\left(\sqrt{x^2-x+2}-x+1\right)}$$

$$\lim_{x \to -1} \frac{1}{(x-1)\left(\sqrt{x^2-x+2}-x+1\right)}$$

$$= -\frac{1}{8}$$

Q5: 7 marks

Find the first and the second derivative of the function

$$f(x) = \left(e^{ax} - e^{-ax}\right) x e^{ax}$$

Also caculate the exact values of $f'(x)$ and $f''(x)$ at $x=0$.

Solution:

$$f(x) = x e^{2ax} - x.$$

$$f'(x) = (x)' e^{2ax} + x \cdot (e^{2ax})' - 1$$

$$= e^{2ax} + x \cdot (e^{2ax}) \cdot 2a - 1$$

$$= (2ax+1) \cdot e^{2ax} - 1$$

$$f''(x) = \left[(2ax+1) \cdot e^{2ax} - 1 \right]'$$

$$= (2ax+1)' \cdot e^{2ax} + (2ax+1) \cdot (e^{2ax})'$$

$$= (2a) \cdot e^{2ax} + (2ax+1) \cdot (e^{2ax}) \cdot (2a)$$

$$= (4a^2x + 4a) e^{2ax}$$

$$f'(0) = 1 - 1 = 0$$

$$f''(0) = 4a$$

→ 8 marks

Q6: Find the derivative of the following functions.

a. $f(x) = \ln[x^2 \arctan(x) + x \cos(x^2)]$

b. $f(x) = (1+x^2)^{\tan(x)}$

Solution:

a). $$f'(x) = \frac{1}{x^2 \arctan(x) + x\cos(x^2)} \left[x^2 \arctan(x) + x\cos(x^2) \right]'$$

$$= \frac{1}{x^2 \arctan(x) + x\cos(x^2)} \left\{ \left[x^2 \arctan(x) \right]' + \left[x\cos(x^2) \right]' \right\}$$

$$= \frac{1}{x^2 \arctan(x) + x\cos(x^2)} \left[(2x)\arctan(x) + \frac{x^2}{1+x^2} + \cos(x^2) - 2x^2 \sin(x^2) \right]$$

$$[x^2 \arctan(x)]' = (x^2)' \arctan(x) + x^2 \cdot [\arctan(x)]'$$
$$= (2x) \cdot \arctan(x) + x^2 \cdot \frac{1}{1+x^2}$$

$$[x \cos(x^2)]' = (x)' \cos(x^2) + x \cdot [\cos(x^2)]'$$
$$= \cos(x^2) + x \cdot [-\sin(x^2)] \cdot 2x$$

b). $y = (1+x^2)^{\tan x}$

take ln( ) of both sides

$$\ln y = (\tan x) \ln(1+x^2)$$

differeniate both sides with respect to x

$$\frac{1}{y}\frac{dy}{dx} = (\sec^2 x) \ln(1+x^2) + [\tan x] \cdot \frac{2x}{1+x^2}$$

$$\frac{dy}{dx} = y \left[(\sec^2 x) \ln(1+x^2) + \frac{2x \tan x}{1+x^2} \right]$$

$$= (1+x^2)^{\tan x} \left[\sec^2 x \cdot \ln(1+x^2) + \frac{2x \tan x}{1+x^2} \right]$$

14 marks

Q7: Consider the function $f(x) = \sqrt{2x+1}$

a). Caculate $f'(x)$ using its definition

b). Write the linear formula for f at $a=4$. and use it to approximate the value of $f(5) = \sqrt{11}$.

Solution:

a). $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{2(x+h)+1} - \sqrt{2x+1}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{2(x+h)+1} - \sqrt{2x+1}}{h} \cdot \frac{\sqrt{2(x+h)+1} + \sqrt{2x+1}}{\sqrt{2(x+h)+1} + \sqrt{2x+1}}$$

$$= \lim_{h \to 0} \frac{2}{\sqrt{2(x+h)+1} - \sqrt{2x+1}}$$

$$= \frac{2}{2\sqrt{2x+1}} = \frac{1}{\sqrt{2x+1}}$$

b). $f(4) = 3$ $\quad (4, 3)$

$f'(4) = \frac{1}{3}$

$y - 3 = \frac{1}{3}(x-4)$

$y = \frac{1}{3}x + \frac{5}{3}$

So $L(x) = \frac{1}{3}x + \frac{5}{3}$ the linear approximation of f at $a=4$

$f(5) = \sqrt{11} \approx 3.317$

$L(5) = \frac{1}{3} \cdot 5 + \frac{5}{3} = \frac{10}{3} \approx 3.33$

Q8: 7 marks.

Consider the curve defined by the equation

$$y^2 + x\sqrt{3+y} = 1 + x^2$$

a). verify that $(2,1)$ belong to this curve and find the implicit derivative $\frac{dy}{dx}$ at this point.

b). write the equation of the tangent line at that point.

Solution:

a) left $= 1^2 + 2\sqrt{3+1} = 5$

right $= 1 + 2^2 = 5$

thus, $(2,1)$ belong to this curve

$$y^2 + x\cdot\sqrt{3+y} = 1 + x^2$$ differentiate both side with x.

$$2y\frac{dy}{dx} + \sqrt{3+y} + x\cdot\frac{1}{2\sqrt{3+y}}\frac{dy}{dx} = 2x$$

$$\frac{dy}{dx} = \frac{2x - \sqrt{3+y}}{2y + \frac{x}{2\sqrt{3+y}}}$$

$$\left.\frac{dy}{dx}\right|_{(2,1)} = \frac{4}{5}$$

b) $y - 1 = \frac{4}{5}(x - 2)$

$\Rightarrow \quad y = \frac{4}{5}x - \frac{3}{5}$

Q9: Using the L-Hospital to evaluate

7 marks

$$\lim_{x \to 0} \frac{\ln(1+x^2)}{\cos(2x)}$$

Solution:

$$\lim_{x \to 0} \frac{\ln(1+x^2)}{\cos(2x)}$$

$\frac{0}{0}$

$$= \lim_{x \to 0} \frac{\frac{\cancel{2}x}{1+x^2}}{\cancel{2}\sin(2x)}$$

$$= \lim_{x \to 0} \frac{\frac{x}{1+x^2}}{\sin(2x)}$$

$\frac{0}{0}$

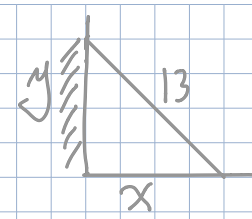
$$= \lim_{x \to 0} \frac{\frac{(1+x^2) - x\cdot(2x)}{(1+x^2)^2}}{2\cos(2x)}$$

$$= \frac{1}{2} = \frac{1}{2}$$

Q10: 7 marks

A 13ft ladder is leaning against a vertical wall of a house when its base starts to slide away (in horizontal direction) from the wall. By the time the base is at the distance $x = 5$ft from the wall, the base is moving at the rate of $\frac{dx}{dt} = 2$ft/sec. How fast is the top of the ladder sliding down the wall at that instant?

Solution: $x^2+y^2=13^2$



differenciate both side with respect to t.

$$2x\frac{dx}{dt}+2y\frac{dy}{dt}=0$$

when $x=5$, $\frac{dx}{dt}=2$ $\quad y=\sqrt{13^2-5^2}=12$

$$2\cdot 5\cdot 2+2\cdot 12\frac{dy}{dt}=0$$

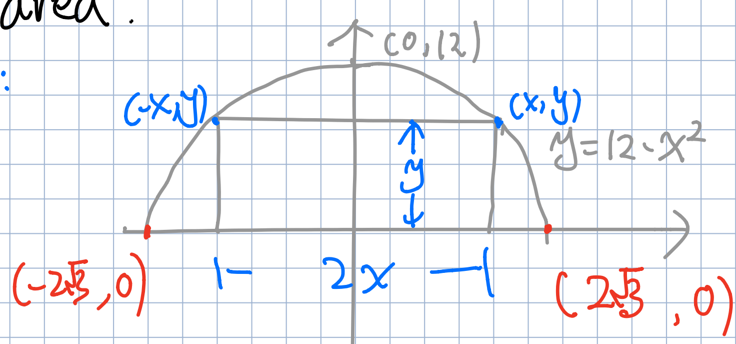
$$\Rightarrow \frac{dy}{dt}=-\frac{5}{6}\ \text{ft/sec}$$

Thus, the top of the ladder is sliding down the wall $\frac{5}{6}$ ft/sec.

7 marks

Q11. A rectangle is inscribed with its base on the x-axis and its upper corners on the parabola $y=12-x^2$.
Find the dimensions of such rectangle with the maximum possible area.

Solution:



$12-x^2=0$

$x=\pm 2\sqrt{3}$

$(2\sqrt{3},0)$

$(-2\sqrt{3},0)$

Base: $2x$

height: y

area $A=2xy$

$\quad =2x\cdot(12-x^2)$

$A = 24x - 2x^3 \quad 0 \leq x \leq 2\sqrt{3}$

$\frac{dA}{dx} = 24 - 6x^2$

$\frac{dA}{dx} = 0 \Rightarrow 24 - 6x^2 = 0 \Rightarrow x = 2$ or $x = -2$

reject.

A(x) achieve the maximum when $x = 2$.

$A(2) = 2 \cdot 2 \cdot (12 - 2^2) = 32$

Thus, the maximum possible area : $\boxed{32}$

the dimention : 4×8

1.5 marks

Q12. Given the function $f(x) = x^4 - 6x^2$.

a. Caculate $f'(x)$ and use it to determine intervals where the function is increasing. intervals where the function is decreasing, and the local extrema (if any)

b. Caculate $f''(x)$ and use it to determine intervals where the function is concave upward, intervals where the function is concave downward. and inflection points (if any)

Solution.

a). $f'(x) = 4x^3 - 12x = 4x(x^2 - 3)$

$f'(x) > 0 \Rightarrow x(x^2 - 3) > 0$

increasing: $(-\sqrt{3}, 0) \cup (\sqrt{3}, +\infty)$

decreasing: $(-\infty, -\sqrt{3}) \cup (0, \sqrt{3})$

$f'(x) = 0 \Rightarrow x=0,\ x=-\sqrt{3},\ x=\sqrt{3}$ critical points

$f'(x)$: $-$ | $-\sqrt{3}$ | $+$ | 0 | $-$ | $\sqrt{3}$ | $+$

$f(0) = 0$

$f(-\sqrt{3}) = -9$

$f(\sqrt{3}) = -9$

local maxi 0 at $x=0$

local mini -9 at $x=-\sqrt{3}$

local mini -9 at $x=\sqrt{3}$

b). $f''(x) = 12x^2 - 12 = 12(x^2-1)$

$+$ | -1 | $-$ | 1 | $+$

$f''(x) > 0 \Rightarrow x < -1$ or $x > 1$

$f''(x) < 0 \Rightarrow -1 < x < 1$

concave up: $(-\infty, -1) \cup (1, +\infty)$

concave down: $(-1, 1)$

inflection points: $x = -1$

$x = 1$