5 Q1: Solve for x :

$$9^x + 4 \cdot 3^x = 12$$

Solution:

$$(3^x)^2 + 4 \cdot 3^x - 12 = 0$$

let $a = 3^x$

$$a^2 + 4a - 12 = 0$$

$$(a-2)(a+6) = 0$$

$a = 2$ or $a = -6$. ✗

$a = 3^x > 0$

$$3^x = 2$$

$$\Rightarrow \quad x = \log_3 2$$

5 Q2: Find the inverse f^{-1} of the formula $f(x) = \log_3 (9 + 3^x)$
Determine the domain and the range of f and f^{-1}.

Solution:

$$y = \log_3 (9 + 3^x)$$

$$3^y = 9 + 3^x$$

$$3^x = 3^y - 9$$

$$x = \log_3 (3^y - 9)$$

so $f^{-1}(x) = \log_3 (3^x - 9)$

	f	f^{-1}
domain :	$(-\infty, +\infty)$	$(2, +\infty)$
range :	$(2, +\infty)$	$(-\infty, +\infty)$

$y = \log_3 (3^x - 9)$

$3^x - 9 > 0 \Rightarrow 3^x > 9 = 3^2 \Rightarrow x > 2$

$y = \log_3 (9 + 3^x)$

$9 + 3^x > 0 \Rightarrow x \in R$

6 Q3: Find all a) horizonal and b) vertical asymptotes of the graph of

$$f(x) = \frac{2x+3}{\sqrt{x^2-2x-3}}.$$

Solution:

1). H.A: $y = 2$. $y = -2$

$$\lim_{x \to +\infty} \frac{2x+3}{\sqrt{x^2-2x-3}} = \lim_{x \to +\infty} \frac{2+\frac{3}{x}}{\sqrt{1-\frac{2}{x}-\frac{3}{x^2}}} = \frac{2}{\sqrt{1}} = 2$$

$$\lim_{x \to -\infty} \frac{2x+3}{\sqrt{x^2-2x-3}} = \lim_{x \to -\infty} \frac{2+\frac{3}{x}}{\frac{\sqrt{x^2-2x-3}}{-\sqrt{x^2}}} = \lim_{x \to -\infty} \frac{2+\frac{3}{x}}{\boxed{-}\sqrt{1-\frac{2}{x}-\frac{3}{x^2}}}$$

$$= \frac{2}{-1} = -2$$

$x \to +\infty$. $x > 0$. $x = \sqrt{x^2}$

$x \to -\infty$ $x < 0$ $x = -\sqrt{x^2}$

2). V.A:

$$\sqrt{x^2-2x-3} = 0$$

$$\Rightarrow \quad x^2-2x-3 = 0$$

$$\Rightarrow \quad (x-3)(x+1) = 0$$

$$x=3. \quad x=-1$$

5 Q4: Find the limit, If the limit does not exist, explain why.

$$\lim_{x\to -2} \frac{x^2-x-6}{|x^2-4|}$$

Solution:

1). $x \to -2^+$: $x > -2$. $x^2 < 4$ $x^2-4 < 0$.

$$|x^2-4| = 4-x^2 = -(x^2-4)$$

$$\lim_{x\to -2^+} \frac{x^2-x-6}{|x^2-4|} = \lim_{x\to -2^+} \frac{x^2-x-6}{-(x^2-4)}$$

$$= \lim_{x\to -2^+} \frac{(x-3)\cancel{(x+2)}}{-(x-2)\cancel{(x+2)}}$$

$$= \frac{(-2-3)}{-(-2-2)} = \frac{-5}{4}$$

2). $x \to 2^-$: $x < -2$ $x^2 > 4$ $x^2-4 > 0$.

$$|x^2-4| = (x^2-4)$$

$$\lim_{x\to -2^-} \frac{x^2-x-6}{|x^2-4|} = \lim_{x\to -2^-} \frac{(x-3)(x+2)}{(x^2-4)}$$

$$= \lim_{x\to -2^-} \frac{(x-3)\cancel{(x+2)}}{(x-2)\cancel{(x+2)}}$$

$$= \frac{-2-3}{(-2-2)} = \frac{-5}{-4} = \frac{5}{4}$$

$$\lim_{x\to -2^-} f(x) \neq \lim_{x\to -2^+} f(x) \quad \text{DNE}$$

4 Q5: Find the limit. If the limit does not exisit, explain why.

$$\lim_{x\to\infty} \left(\sqrt{x^2+4} - x\right)$$

Solution:

$$\left(\sqrt{x^2+4} - x\right) \cdot \frac{\sqrt{x^2+4} + x}{\sqrt{x^2+4} + x}$$

$$= \frac{x^2+4-x^2}{\sqrt{x^2+4} + x}$$

$$= \frac{4}{\sqrt{x^2+4} + x}$$

$$\lim_{x\to\infty} \left(\sqrt{x^2+4} - x\right)$$

$$= \lim_{x\to\infty} \frac{4}{\sqrt{x^2+4} + x}$$

$$= \lim_{x\to\infty} \frac{\frac{4}{x}}{\sqrt{1+\frac{4}{x^2}} + 1}$$

$$= \frac{0}{1+1} = 0$$

6 Q6. Find the second derivative of the function

$$f(x) = \frac{1-\cos x}{1+\cos x}$$

and $f''(0) = ?$

Solution:

$$f(x) = \frac{-(1+\cos x)+2}{1+\cos x} = -1 + \frac{2}{1+\cos x}$$

$$f'(x) = \left[-1 + \frac{2}{1+\cos x}\right]'$$

$$= 2\left[(1+\cos x)^{-1}\right]'$$

$$= 2\cdot(-1)\cdot(1+\cos x)^{-2}\cdot(1+\cos x)'$$

$$= (-2)\cdot(1+\cos x)^{-2}\cdot(-\sin x)$$

$$= 2\sin x\,(1+\cos x)^{-2} = \frac{2\sin x}{(1+\cos x)^2}$$

$f''(x) = 2\left[\sin x\,(1+\cos x)^{-2}\right]'$ by product rule

$$= 2\left\{(\sin x)'(1+\cos x)^{-2} + \sin x\left[(1+\cos x)^{-2}\right]'\right\}$$

$$= 2\left[\cos x\,(1+\cos x)^{-2} + \sin x\,(-2)\cdot(1+\cos x)^{-3}\cdot\underbrace{(1+\cos x)'}_{-\sin x}\right]$$

$$= 2\left[\cos x(1+\cos x)^{-2} + 2(\sin x)^2(1+\cos x)^{-3}\right]$$

$$f'(0) = 2\cdot 1\cdot(1+1)^{-2} = \frac{2}{2^2} = \frac{1}{2}$$

$\sin 0 = 0$ $\quad \cos 0 = 1$

6 Q7: Find the derivative of the following functions.

a). $f(x) = \dfrac{x^2 e^{-x}}{1+e^x}$

b). $f(x) = \sin(\underbrace{x^2\sin^2 x + x^2 - \cos^2 x}_{u})$

Solution:

a)

$$f'(x) = \frac{(x^2e^{-x})'(1+e^x) - x^2e^{-x}(1+e^x)'}{[1+e^x]^2}$$ Quotient rule

$$= \frac{(2xe^{-x} - x^2e^{-x})(1+e^x) - x^2\boxed{e^{-x}\,e^x}}{(1+e^x)^2}$$ $e^0 = 1$

$$= \frac{(2x-x^2)(1+e^{-x}) - x^2}{(1+e^x)^2}$$

b)

$$f'(x) = \cos(x^2\sin^2 x + x^2 - \cos^2 x)\cdot(x^2\sin^2 x + x^2 - \cos^2 x)'$$

$$= \cos(x^2\sin^2 x + x^2 - \cos^2 x)\cdot(2x\sin^2 x + 2x^2\sin x\cos x + 2x + 2\cos x\sin x)$$

$$= \cos(x^2\sin^2 x + x^2 - \cos^2 x)\left[2x\sin^2 x + (2x^2+2)\sin x\cos x + 2x\right]$$

$$(x^2 \sin^2 x)' = (x^2)' \sin^2 x + x^2 (\sin^2 x)'$$
$$= 2x \sin^2 x + x^2 \cdot 2\sin x (\cos x)$$

6 Q8: Caculate $f'(x)$ using definition

$$f(x) = \sqrt{1+2x}$$

Solution:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{1+2(x+h)} - \sqrt{1+2x}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{1+2(x+h)} - \sqrt{1+2x}}{h} \cdot \frac{\sqrt{1+2(x+h)} + \sqrt{1+2x}}{\sqrt{1+2(x+h)} + \sqrt{1+2x}}$$

$$= \lim_{h \to 0} \frac{2\cancel{h}}{\cancel{h} \cdot \sqrt{1+2(x+h)} + \sqrt{1+2x}}$$

$$= \frac{2}{2\sqrt{1+2x}} = \frac{1}{\sqrt{1+2x}}$$

7 Q9:

$$f(x) = \begin{cases} ax+1 & \text{if } x \le 1 \\ x^2+3 & \text{if } x > 1 \end{cases}$$

a. find the value of a that makes $f(x)$ continuous everywhere.

b. Is the function $f(x)$ also differeniable everywhere? explain why yes or why not.

Solution:

a). $f(x)$ is continuous everywhere

$\Rightarrow$ $f(x)$ is continuous at $x=1$.

$$\lim_{x\to 1} f(x) = f(1) \qquad f(1) = a+1$$

$$\Rightarrow \lim_{x\to 1^+} f(x) = f(1)$$

$$\Rightarrow \lim_{x\to 1^+} (x^2+3) = a+1$$

$$1^2+3 = a+1$$

$$\Rightarrow a=3$$

2).

$$f(x) = \begin{cases} 3x+1 & \text{if } x\leq 1. \\ x^2+3 & \text{if } x>1. \end{cases}$$

$f(x)$ is not differenitable at $x=1$.

$$f'(1) = \underline{\lim_{h\to 0} \frac{f(1+h)-f(1)}{h}}$$ limit does not exist.

?

$h\to 0^+$, $h>0$, $1+h>1$. $f(1+h) = (1+h)^2+3$

$h\to 0^-$, $h<0$, $1+h<1$. $f(1+h) = 3(1+h)+1$

$f(1) = 4$.

$$\lim_{h\to 0^+} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^+} \frac{(1+h)^2+3-4}{h}$$

$$= \lim_{h\to 0^+} (h+2) = 2$$

$$\lim_{h\to 0^-} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^-} \frac{3(1+h)+1-4}{h}$$

$$= \lim_{h\to 0^-} \frac{3h}{h} = 3$$

$$\lim_{h\to 0^+} \frac{f(1+h)-f(1)}{h} \neq \lim_{h\to 0^-} \frac{f(1+h)-f(1)}{h}$$

$$\Rightarrow \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} \quad \text{DNE}$$

$\Rightarrow$ $f(x)$ is not differentiable at $x=1$.