Question 1: Solve the initial value problem

$$xyy' = 2e^{-y^2}\ln x; \quad y(1) = 2$$

ANS:

$$xyy' = 2e^{-y^2}\ln x$$

$$\Rightarrow e^{y^2}yy' = 2\frac{\ln x}{x}$$

$$\Rightarrow e^{y^2}ydy = 2\frac{\ln x}{x}dx$$

Let $y^2 = u \Rightarrow 2ydy = du$.
Let $\ln x = v \Rightarrow \frac{1}{x} = dv$.
By substitution, we get

$$\frac{e^u}{2}du = 2vdv$$

$$\Rightarrow \int \frac{e^u}{2}\, du = 2\int v\ dv$$

$$\Rightarrow \frac{e^u}{2} = v^2 + c_1$$

$$\Rightarrow \frac{e^{y^2}}{2} = (\ln x)^2 + c_1$$

$$\Rightarrow e^{y^2} = 2(\ln x)^2 + c$$

Using initial condition, $y(1) = 2$

$$e^{2^2} = 2(\ln 1)^2 + c \Rightarrow c = e^4$$

So the particular solution is

$$e^{y^2} = 2(\ln x)^2 + e^4$$

$$y^2 = \ln\left(2(\ln x)^2 + e^4\right)$$

$$y = \pm\sqrt{\ln\left(2(\ln x)^2 + e^4\right)}$$

Question 2:

a) Find the general solution of the equation

$$xy' = 1 + y^2$$

ANS:

$$xy' = 1 + y^2$$

$$\Rightarrow \frac{1}{1+y^2}y' = \frac{1}{x}$$

$$\Rightarrow \frac{1}{1+y^2}dy = \frac{1}{x}dx$$

$$\Rightarrow \int \frac{1}{1+y^2}\,dy = \int \frac{1}{x}\,dx$$

$$\Rightarrow \tan^{-1}y = \ln(x) + c$$

$$\Rightarrow y = \tan(\ln(x) + c)$$

b) Using the initial conditions $y(1) = 1$, find the particular solution.

ANS: Using initial condition, $y(1) = 1$,

$$1 = \tan(\ln(1) + c) \Rightarrow \tan c = 1 \Rightarrow c = \frac{\pi}{4}$$

So the particular solution is

$$y = \tan\left(\ln(x) + \frac{\pi}{4}\right)$$

c) Find the orthogonal trajectory of one parameter family found in **a)**.

ANS: To find the orthogonal trajectory, we need to solve

$$x\left(-\frac{1}{y'}\right) = 1 + y^2$$

$$\Rightarrow (1 + y^2)y' = -x$$

$$\Rightarrow \int (1 + y^2)\, dy = \int -x dx$$

$$\Rightarrow y + \frac{y^3}{3} = -\frac{x^2}{2} + c$$