MAT 1322 B Winter 2018 February 7th, 8:30 Elisabeth Fink

Midterm 1 Solutions

Max. points = 22

Last Name: ______________________ **First Name:** ______________________

Student Number: ______________________

DGD:	Yves	Nicole	Stephanie	Blair
	10:00 (VNR 3075)	11:30 (FTX 359)	13:00 (FTX 351)	14:30 (SMD 422)

- Length: 80 min.
- It is a closed book exam.
- You have to answer all questions.
- You have to justify all your answers with a clear and complete solution.
- Answer directly on the exam and you can use the back of each page for your preliminary computations. **Scrap paper is forbidden**.
- **The Faculty approved calculators may be used. All others are forbidden.**
- Cellular phones and unauthorized electronic devices are not allowed during this exam. Phones and devices must be turned off and put away in your bag. Do not keep them in your possession, such as in your pockets. If caught with such a device or document, the following may occur: academic fraud allegations will be filed which may result in you obtaining a 0 (zero) for the exam.

By signing below, you acknowledge that you have ensured that you are complying with the above statement.

Signature : ______________________

Multiple Choice Answers:

☐	☐	☐	☐
#1	#2	#3	#4

Multiple-choice Questions: No justification is required.

1. (2 pts) What is the area of the region bounded by the curves $y = 2x^2$ and $x+y = 1$ for $-1 \leq x \leq \frac{1}{2}$?

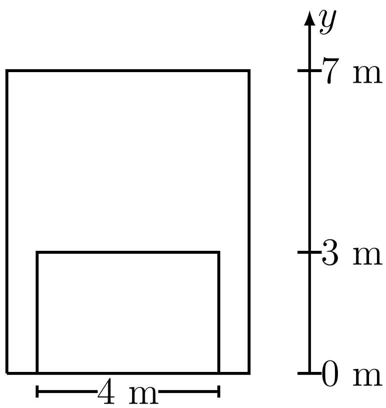
A. $\frac{8}{5}$ **B.** $\frac{7}{8}$ **C.** $\frac{9}{8}$ **D.** $\frac{1}{4}$ **E.** $\frac{3}{2}$ **F.** $\frac{1}{5}$

G. None of the above.

The second curve can be described by $y = 1 - x$. We compute the following integral:

$$A = \int_{-1}^{\frac{1}{2}} (1-x) - 2x^2 dx = \frac{9}{8}.$$

2. (2 pts) A reservoir has a rectangular door located at the bottom of one of its vertical sides, as shown in the diagram below. The reservoir is 7 m high and filled to the top with water. The door is 4 m wide by 3 m high.



Let y represent the height from the bottom of the reservoir. Which of the following integrals represents the hydrostatic force exerted by the water on the door? Note that the density of water is 1000 kg/m^3 and the acceleration due to gravity is 9.8 m/s^2.

A. $9800 \int_3^7 6(y-7)dy$ **B.** $9800 \int_0^3 12y dy$ **C.** $9800 \int_0^3 4(7-y)dy$

D. $9800 \int_0^7 12(y-7)dy$ **E.** $9800 \int_0^3 2(y-7)dy$ **F.** $9800 \int_0^7 84 dy$

G. None of the above.

Solution: First, we note that the object in question ranges from height $y = 0$ to $y = 3$. Hence only integrals with bounds 0 and 3 can be the right ones. Further, the width of the object is 4 m, so we need the factor 4. Finally, the total depth of the water is 7 m, so the depth of our object is at the highest point $7 - 3$ meters and at the lowest point 7 m. We get an additional factor $7 - y$ from this. Hence the right answer is C.

3. (2 pts) What is the arc length of the curve $y = 2x^{\frac{3}{2}} - 1$ between $x = 0$ and $x = 9$, rounded to one decimal place?

A. 156.1 **B.** 25.2 **C.** 51.7 **D.** 74.4 **E.** 14.3 **F.** 54.9

G. None of the above.

Solution: For the arc length we have the formula $L = \int_a^b \sqrt{1 + f'(x)}dx$. We hence first find the first derivative $y'(x) = 3\sqrt{x}$. Then we compute

$$L = \int_0^9 \sqrt{1 + (3\sqrt{x})^2}dx = 54.9$$

4. (2 pts) Let $\mathcal{R}$ be the region bounded by the curve $y = e^{3x}$, the x-axis, and the lines $x = 2$ and $x = 4$.

Which of the following integrals represents the volume of the solid obtained by rotating the region $\mathcal{R}$ about the x-axis ?

A. $\int_2^4 \pi e^{3x}dx$ **B.** $\int_2^4 2\pi e^{3x}dx$ **C.** $\int_2^4 \pi e^{6x}dx$

D. $\int_2^4 2\pi e^{6x}dx$ **E.** $\int_0^e \pi e^{3x}dx$ **F.** $\int_0^e \pi \ln(y/3)dy$

G. None of the above.

Solution: We rotate a curve around the x-axis, so the cross-sections have the shape of circles. Hence the correct formula requires the cross-section area function $A(x) = r(x)^2 \cdot \pi$. The bounds are 2 and 4, so we can immediately exclude integrals with other bounds. Our function is $f(x) = e^{3x}$, which represents the radius. Hence the correct integral is $\int_2^4 (e^{3x})^2 \cdot \pi$, which is answer C.

5. (2 pts) Use Euler's method with step size $h = 0.2$ to estimate $y(0.4)$, where $y(x)$ is the solution to the differential equation $y' = x - y^2$ with initial condition $y(0) = -1$.

A. -0.972 **B.** -1.2 **C.** 0.833 **D.** -1.422 **E.** -1.56 **F.** -1.448

G. None of the above.

Solution: Euler's formula uses $y_0 = -1, x_0 = 0$ and $F(x, y) = x - y^2$ here. We obtain $y_1 = -1.2$ as approximation of $y(0.2)$ and $y_2 = -1.448$, which approximates $y(0.4)$.

LONG-ANSWER QUESTIONS: Give detailed solutions, clearly showing each of your steps.

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6. Consider the following differential equation: $\frac{dy}{dx} + 4xy^3 = 0$

(a) (2 pts) Determine its general solution.

Solution:

$$\frac{dy}{dx} = -4xy^2$$
$$\frac{dy}{y^4} = -4xdx$$
$$-\frac{1}{3y^3} = -2x^2 + C$$
$$y = \sqrt[3]{\frac{1}{6x^2 - C}}.$$

(b) (1 pt) Determine the particular solution for which $y(0) = 5$.

Solution: We need to find C and express the function with the new value:

$$5 = \sqrt[3]{\frac{1}{6 \cdot 0^2 - C}}$$
$$125 = \frac{1}{-C}$$
$$C = -\frac{1}{125}$$

Hence the particular solution for the given initial value is:

$$y = \sqrt[3]{\frac{1}{6x^2 + \frac{1}{125}}}.$$

7. Improper Integrals.

(a) (2 pts) Consider the integral $\displaystyle\int_0^2 \frac{\ln x}{x}\, dx\,.$

Is it an improper integral (**justify**)? Does it converge? If yes, then compute it.

Solution:

The integral is improper because the integrand is not defined at the lower bound $x = 0$.

For computing it, we replace the lower bound by t. We proceed by substitution $u = \ln(x), du = \frac{1}{x}dx$:

$$\int_t^2 \frac{\ln(x)}{x}\, dx = \int_{\ln(t)}^{\ln(1)} u\, du = \frac{u^2}{2}|_{\ln(t)}^0 = -\frac{\ln(t)^2}{2}.$$

Now we take the limit $t \to 0$:

$$\int_0^2 \frac{\ln x}{x}\, dx = \lim_{t\to 0}(-\frac{\ln(t)^2}{2}) = -\infty.$$

Hence the integral diverges.

(b) (2 pts) Using the comparison test, determine whether the following improper integral converges or not:

$$\int_1^\infty \frac{\arctan(x)}{x^2+e^x}\, dx$$

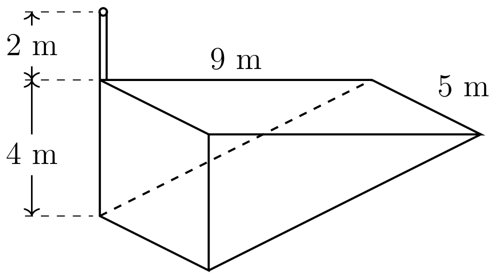
Solution: We note that $-\frac{\pi}{2} \leq \arctan(x) \leq \frac{\pi}{2}$. Hence

$$\int_1^\infty \frac{\arctan(x)}{x^2+e^x}\, dx \leq \int_1^\infty \frac{\frac{\pi}{2}}{x^2+e^x}\, dx = \frac{\pi}{2} \cdot \int_1^\infty \frac{1}{x^2+e^x}\, dx \leq \frac{\pi}{2}\int_1^\infty \frac{1}{x^2}\, dx$$

because $x^2 + e^x > x^2$ on $[1, \infty)$. The last integral is of the known form $\int_1^\infty \frac{1}{x^p}\, dx$, which is known to converge for $p > 1$. Hence the original improper integral converges.

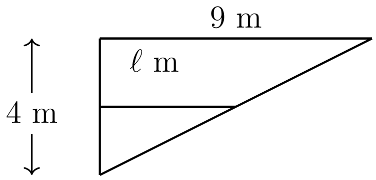
8. (5 pts) A pool as shown in the picture is filled with water. What work is done by pumping the water 2 m above the top of the pool? Clearly define all variables that enter into your solution and provide a diagram which shows their meaning.

Note that the density of water is 1000 kg/m^3 and the acceleration due to gravity is 9.8 m/s^2.



Solution:

Let x denote the distance of the i-th slice to the surface of the pool. Looking at the pool from the side we get the following picture:



First, by similar triangles, we find the width of such a slice, denoted by $\ell(x)$:

$$\frac{\ell(x)}{4-x} = \frac{9}{4} \Rightarrow \ell(x) = \frac{9}{4}(4-x) = 9 - \frac{9}{4}x.$$

The cross-section area of the pool at depth x is

$$A(x) = 5\ell(x_i) = 5\left(9 - \frac{9}{4}x\right) = 45 - \frac{45}{4}x.$$

The volume of a thin slice of water of width Δx is

$$V(x, \Delta x) = \ell(x)\Delta x = \left(45 - \frac{45}{4}x\right)\Delta x.$$

Then, the mass of a thin slice of water of width Δx is

$$m(x, \Delta x) = \rho(45 - \frac{45}{4}x)\Delta x.$$

Hence the force required to raise this layer of water of width Δx is

$$F(x, \Delta x) = \rho g \left(45 - \frac{45}{4}x\right)\Delta x.$$

We want to pump this slice $2 + x$ meters above its position, the work required to do that is:

$$W(x, \Delta x) = \rho g \cdot \left(45 - \frac{45}{4}x\right)(2 + x)\Delta x.$$

The work required to pump all the water from the pool is hence the sum over all such slices, where now the Δx becomes dx in the limit when we let the height of a slice go to 0.

$$W = \int_0^4 \rho g \cdot \left(45 - \frac{45}{4}x\right)(2+x)dx = 2940000\,J$$

This integral has the standard form as learned in class:

$$W = \int_a^b \rho g \cdot A(x) \cdot x\,dx,$$

where we here denote by x the distance from the top of the hose and $A(x)$ is the cross-section area when we are x meters from the top of the hose. The bounds a and b are then the minimal lifting distance and the maximal lifting distance. Hence an equivalent integral would be

$$W = \int_2^6 \rho g x \cdot \frac{9}{4}(6-x) \cdot 5 = 2940000\,J,$$

where here $A(x) = \frac{9}{4}(6-x) \cdot 5$ is the cross-section area and we simply have x as lifiting distance.