Université d'Ottawa · University of Ottawa
Faculté des sciences
Mathématiques et de statistique
Faculty of Science
Mathematics and Statistics

**MAT 1341C — Second Midterm Test (V4)** **Instructor: Mateja Šajna**

Last name: ____________________________________

First name: ____________________________________

Student number: __________________ Signature: __________________

*Circle:* DGD1 (MRT 250) DGD2 (CBY B012) DGD3 (STE G0103) DGD4 (TBT 325)

**Instructions:**

- This is an 80-minute **closed-book** exam; **no notes** are permitted. Only **calculators without a SIN button** are permitted.
- The test consists of 7 questions on ?? pages. Page ?? gives additional workspace – please do not detach it. Use the backs of the pages for rough work. Do not use scrap paper of your own.
- Carefully read the instructions for each question. In multiple-choice questions, **cross the box in front of the correct answer**. In true/false questions, write the appropriate response in the answer box. Fully justify your answers where requested. You may ask for clarification.
- Unauthorized electronic devices (such as cellular phones) are not permitted during this test. Such devices must be **turned off completely** and stored out of students' reach (not in pockets). Academic fraud allegations may be filed if a student is found in possession of such a device during the test.

*For markers only — do not write in the table below.*

| **Question** | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **Total** |
|---|---|---|---|---|---|---|---|---|
| **Mark** | | | | | | | | |
| **Max** | 1 | 1 | 1 | 1 | 1 | 5 | 6 | 16 |

**1.** Which of the following are subspaces of $\mathbb{R}^3$? [1pt]

$$U = \{(x, y, z) \in \mathbb{R}^3 : x + 3y - 4z = -1\}$$

$$V = \{(x, y, z) \in \mathbb{R}^3 : 2x - 3y + 6z = 0\}$$

$$W = \{(x^2, y^2, z) \in \mathbb{R}^3 : x, y, z \in \mathbb{R}\}$$

$$X = \{(x + y, 4y, -2x + y) \in \mathbb{R}^3 : x, y \in \mathbb{R}\}$$

*Cross (X) the box for the correct answer:*

- [A] Only $W$ and $X$
- [B] Only $U$ and $V$
- [C] Only $U$ and $W$
- [D] Only $U$ and $X$
- [E] Only $V$ and $W$
- [F] Only $V$ and $X$ — **Correct**

**Solution:** $U$ does not contain the zero vector, so it is NOT a subspace.

$V$ is a plane through the origin, so it is a subspace.

$W$ is not closed under scalar multiplication (for example, $(1, 0, 0)$ is in $W$, but $(-1)(1, 0, 0)$ is not), so it is NOT a subspace.

$X = \text{span}\{(1, 0, -2), (1, 4, 1)\}$, so $X$ is a subspace.

**2.** Which one of the following is a subspace of the vector space $M_{22}(\mathbb{R})$? [1pt]

*Cross (X) the box for the correct answer:*

A $\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : \ cd = 0, \ a, b, c, d \in \mathbb{R} \right\}$

B $\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : \ a + c + d = -1, \ a, b, c, d \in \mathbb{R} \right\}$

C $\left\{ \begin{bmatrix} a & 3 \\ b & b \end{bmatrix} : \ 2a + b = 0, \ a, b \in \mathbb{R} \right\}$

D $\left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : \ a + b - c = 0, \ a, b, c \in \mathbb{R} \right\}$ — **Correct**

E $\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : \ cd = 1, \ a, b, c, d \in \mathbb{R} \right\}$

F None of the above

**Solution:** The zero-matrix does not belong to any of C, B, and E. So those are NOT subspaces. A is not closed under addition: matrices $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ both belong to A, but their sum does not. D is closed under addition and multiplication by scalars, so it is a subspace.

**3.** For what value of $t$ is the set of vectors $\{(1,1,1),(-1,1,2),(0,t,6)\}$ linearly **dependent**?

[1pt]

*Cross (X) the box for the correct answer:*

- [A] 5
- [B] 4 — **Correct**
- [C] 3
- [D] 7
- [E] 6
- [F] 2

**Solution:** Since the set $\{(1,1,1),(-1,1,2)\}$ is linearly independent, for $\{(1,1,1),(-1,1,2),(0,t,6)\}$ to be linearly dependent, the vector $(0,t,6)$ must be a linear combination of $(1,1,1)$ and $(-1,1,2)$. We can either guess $2\cdot(1,1,1)+2\cdot(-1,1,2)=(0,4,6)$, or else set up the system

$$x(1,1,1)+y(-1,1,2)=(0,t,6).$$

We can then show that this system is consistent if and only if $t=4$. So the correct answer is 4.

**4.** Let $\{u, v, w\}$ be a linearly **independent** set in a vector space $V$, and suppose that vector $z \in V$ is chosen so that the set $\{u, v, w, z\}$ is linearly **dependent**. Which of the statements below is ALWAYS true?

[1pt]

*Cross (X) the box for the correct answer:*

- [A] $z \notin \text{span}\{u, v, w\}$
- [B] $v \in \text{span}\{u, w, z\}$
- [C] $\{u, v, z\}$ is linearly dependent
- [D] $z \in \text{span}\{u, v, w\}$ — **Correct**
- [E] $\{w, z\}$ is linearly dependent
- [F] $V = \text{span}\{u, v, w, z\}$

**Solution:** Since $\{u, v, w, z\}$ is LD and $\{u, v, w\}$ is LI, it follows (from a theorem) that $z \in \text{span}\{u, v, w\}$. Hence [D] is true and [A] is false.

However, $z$ need not be in $\text{span}\{u, v\}$ or $\text{span}\{w\}$, so the sets $\{u, v, z\}$ and $\{w, z\}$ need not be LD. Hence [C] and [E] are false.

Since $\{u, v, w, z\}$ is LD, one of these vectors is in the span of the others, but it need not be $v$, so [B] is false.

[F] is false since $V$ may have vectors not in $\text{span}\{u, v, w, z\}$.

**5.** Let $U$ be a subspace of $\mathbb{R}^{19}$. Suppose $U$ has a linearly dependent set of 12 vectors, and also a linearly independent set of 7 vectors. Then for such $U$ it is ALWAYS true that

[1pt]

*Cross (X) the box for the correct answer:*

- [A] $\dim U \geq 12$
- [B] $\dim U \geq 19$
- [C] $7 \leq \dim U \leq 19$ — **Correct**
- [D] $\dim U \leq 7$
- [E] $7 \leq \dim U \leq 12$
- [F] $12 \leq \dim U \leq 19$

**Solution:** The dimension of $U$ is at least as large as the size of any LI set (so at least 7) and no larger than the dimension of $\mathbb{R}^{19}$ (so at most 19).

Note that any non-trivial vector space has LD sets of any (positive) size, so the assumption that $U$ has a LD set of 12 vectors is irrelevant.

**6.** Consider the vector space $\mathbf{P}_2 = \{a + bx + cx^2 : a, b, c \in \mathbb{R}\}$ of polynomial functions of degree at most 2. Let $U$ be the subset of $\mathbf{P}_2$ defined by

$$U = \{f \in \mathbf{P}_2 : f(5) = 0\}.$$

Thus, the subset $U$ consists of all polynomials of degree at most 2 that have 5 as a root.

a) Show that $U$ is a subspace using the **Subspace Test**. [2pts]

**Solution:** First, we verify that the zero vector (function $f_0$ such that $f_0(x) = 0$ for all $x \in \mathbb{R}$) is in $U$. Indeed, $f_0(5) = 0$, so $f_0 \in U$.

Next, we show closure under addition. Take any polynomials $f, g \in U$. Then $f(5) = 0$ and $g(5) = 0$, so

$$(f + g)(5) = f(5) + g(5) = 0 + 0 = 0.$$

Hence $f + g \in U$.

Finally, we show closure under scalar multiplication. Take any $f \in U$ (so that $f(5) = 0$) and any scalar $c \in \mathbb{R}$. Then

$$(cf)(5) = cf(5) = c \cdot 0 = 0.$$

Hence, $cf \in U$.

b) Show that $U = \operatorname{span}\{5 - x, 5x - x^2\}$ [2pts]

**Solution:** Take any polynomial $f \in U$. We want to show that

$$f = k(5 - x) + \ell(5x - x^2) \qquad \text{for some } k, \ell \in \mathbb{R}.$$

Since $f \in U$, we have $f(x) = a + bx + cx^2$ for some $a, b, c, \in \mathbb{R}$, and also $f(5) = 0$. Therefore, $f(5) = a + 5b + 25c = 0$ and $c = -\frac{a}{25} - \frac{b}{5}$. Substituting we obtain

$$f(x) = a + bx - (\tfrac{a}{25} + \tfrac{b}{5})x^2 = \tfrac{a}{5}(5 - x) + (\tfrac{a}{25} + \tfrac{b}{5})(5x - x^2).$$

Hence, we can take $k = \frac{a}{5}$ and $\ell = \frac{a}{25} + \frac{b}{5}$, and conclude that $f \in \operatorname{span}\{5 - x, 5x - x^2\}$.

Since $f$ was arbitrary, it follows that $U \subseteq \operatorname{span}\{5 - x, 5x - x^2\}$.

Conversely, since $5 - x$ and $5x - x^2$ are in $U$ (as both have 5 as a root), it follows (from a theorem proved in class) that $\operatorname{span}\{5 - x, 5x - x^2\} \subseteq U$.

Hence $U = \operatorname{span}\{5 - x, 5x - x^2\}$.

c) Explain why $\{5 - x, 5x - x^2\}$ is a basis for $U$. [1pt]

**Solution:** From (b) we know that $\{5 - x, 5x - x^2\}$ is a spanning set for $U$. So it only remains to show that this set is linearly independent.

Indeed, if $k(5 - x) + \ell(5x - x^2) = 0$, then (after multiplying out) we obtain

$$5k - kx + 5\ell x - \ell x^2 = 5k + (5\ell - k)x - \ell x^2 = 0,$$

which implies $5k = 5\ell - k = \ell = 0$, so that $k = \ell = 0$.

**7.** For each of the following statements, determine whether it is always true **(T)**, or is possibly false **(F)**. Write your answer in the box following the statement.

*Justify each answer:*

- *If you claim that the statement may be false, you must either give an explicit example where it fails (a counterexample), or explain clearly why it fails (e.g. referring to theorems from class).*
- *If you claim that the statement is always true, you must give a clear explanation.*

a) Let $M_{44}(\mathbb{R})$ be the vector space of $4 \times 4$ matrices with real entries. Then $\dim M_{44}(\mathbb{R}) = 4$.

Answer (T/F): [ ] [1/2pts]

Justification: [1pt]

**Solution:** False. The dimension is $4^2$. (Consider the standard basis consisting of $4 \times 4$ matrices with exactly one non-zero entry, and this entry is 1. There are $4^2$ such matrices.)

b) Let $U = \operatorname{span}\{4, \sin^2(4x), \cos^2(4x)\}$ be a subspace of the vector space of real-valued functions $\mathcal{F}(\mathbb{R})$. Then $\dim(U) = 3$.

Answer (T/F): [ ] [1/2pts]

Justification: [1pt]

**Solution:** False.

As $4 = 4\sin^2(4x) + 4\cos^2(4x)$, the set $\{4, \sin^2(4x), \cos^2(4x)\}$ is linearly dependent. Hence, $\dim(U) < 3$.

c) The vector space of real-valued functions $\mathcal{F}(\mathbb{R})$ contains 5 linearly independent vectors.

Answer (T/F): [1/2pts]

Justification: [1pt]

**Solution:** True.

For example, take the set $\{1, x, \ldots, x^4\}$ of polynomial functions. This set is linearly independent.

d) If $U$ and $W$ are subspaces of a vector space $V$, then the union

$$U \cup W = \{v \in V : v \in U \text{ or } v \in W\}$$

is also a subspace.

Answer (T/F): [1/2pts]

Justification: [1pt]

**Solution:** False.

For example, take $V = \mathbb{R}^2$, $U = \{(x, 0) : x \in \mathbb{R}\}$, and $W = \{(0, y) : y \in \mathbb{R}\}$. Then $(1, 0) \in U \cup W$, $(0, 1) \in U \cup W$, but $(1, 0) + (0, 1) = (1, 1) \notin U \cup W$. Hence $U \cup W$ is not closed under addition, and hence is not a subspace of $V$.

*More space for work. Please do not detach this page.*