



# Université d'Ottawa • University of Ottawa

Faculté des sciences  
Mathématiques et de statistique

Faculty of Science  
Mathematics and Statistics

MAT 1341A – The Midterm Test I (v.2)

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Last name: \_\_\_\_\_

First name: \_\_\_\_\_

Student number:

Please, read the following instructions carefully:

- You have 80 minutes to complete this test. **Do not detach** the pages of this examination. Read each question carefully. Where it is possible to check your work, do so.
- You can use the backs of the pages and the last page for computations.
- This is a closed book exam, and no notes of any kind are allowed. The use of programmable calculators, cell phones, laptops, pagers or any text storage or communication device is not permitted.

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THIS SPACE IS RESERVED FOR THE MARKER:

Question	1	2	3	4	5	6	7	8	Total
Mark									
Out of	1	1	1	1	5	4	1	1	15

1. Find all vectors in  $\mathbb{R}^3$  which are perpendicular to both  $(-2, 2, 10)$  and  $(2, 1, 2)$ . (2)

cross (X) the correct answer:

A  $(t + 6, -4, t + 6) \mid t \in \mathbb{R}$

B  $(1, -4, 2)$

C  $(t, -4t, t) \mid t \in \mathbb{R}$

D  $(-2t, 0, t) \mid t \in \mathbb{R}$

E  $(-1, -1, 1)$

F  $(6, 24, 6)$

Such vectors will be parallel to

$$u \times v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 2 & 10 \\ 2 & 1 & 2 \end{vmatrix} = (-6, 24, -6).$$

Hence  C is correct

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2. If  $u = (2, -1, 1)$  and  $v = (1, 1, 2)$  then the length of the projection of  $u$  along  $v$  is: (1)

cross (X) the correct answer:

A  $\frac{3\sqrt{2}}{2}$

B  $\frac{3\sqrt{6}}{2}$

C  $\frac{2\sqrt{6}}{3}$

D  $\frac{\sqrt{6}}{2}$

E 0

F  $\frac{2\sqrt{2}}{3}$

Solution: Since  $\|proj_v u\| = \frac{|u \cdot v|}{\|v\|^2} \|v\| = \frac{|u \cdot v|}{\|v\|} = \frac{3}{\sqrt{6}} = \frac{\sqrt{6}}{2}$ . So the correct answer is  D.

3. Mark whether each of the following statements is TRUE or FALSE in the respective box.  
(each correct answer is 1/4pt)

- It is possible that a system of linear equations with coefficients in  $\mathbb{R}$  has exactly 3 solutions.

ANSWER:  FALSE

- A homogeneous system of linear equations is always consistent

ANSWER:  TRUE

- There exists a linear system of four equations of rank 6

ANSWER:  FALSE

- A homogeneous system of linear equations can have a unique solution

ANSWER:  TRUE

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4. If the coefficient matrix  $A$  in a homogeneous system in 19 variables of 16 equations is known to have rank 9, how many parameters are there in the general solution? (1)

cross (X) the correct answer:

A 19

B 7

C 9

D 3

E 10

F 16

Solution: Homogeneous system is consistent. So the number of parameters is  $19 - 9 = 10$ . So the correct answer is  E.

5. Suppose  $e, f \in \mathbb{R}$  and consider the linear system in  $x, y$  and  $z$ :

$$\begin{cases} 2x - 2y - ez & = f \\ x + z & = -1 \\ 3x + y + 2z & = -1 \end{cases}$$

5(a) If  $(A \mid b)$  is the augmented matrix of the system above, find the rank of  $A$  and the rank of  $(A \mid b)$  for **all** values of  $e$  and  $f$ . (2)

Solution: We have

$$(A \mid b) = \left( \begin{array}{ccc|c} 2 & -2 & -e & f \\ 1 & 0 & 1 & -1 \\ 3 & 1 & 2 & -1 \end{array} \right) \xrightarrow{\text{Gauss elimination}} \left( \begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & -2 & -e-2 & f+2 \end{array} \right)$$
$$\xrightarrow{\text{Gauss elimination}} \left( \begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & -e-4 & f+6 \end{array} \right)$$

Hence,

$$\text{rank } A = \begin{cases} 2 & \text{if } e = -4 \\ 3 & \text{if } e \neq -4 \end{cases}$$
$$\text{rank } (A \mid b) = \begin{cases} 2 & \text{if } e = -4 \text{ and } f = -6 \\ 3 & \text{otherwise} \end{cases}$$

**5(b)** Using part (a), find all values of  $e$  and  $f$  so that this system has

(i) a unique solution

(1)

Solution: it has a unique solution  $\iff \text{rank } A = \text{rank } (A | b) = 3 \iff e \neq -4$ .

(ii) infinitely many solutions

(1)

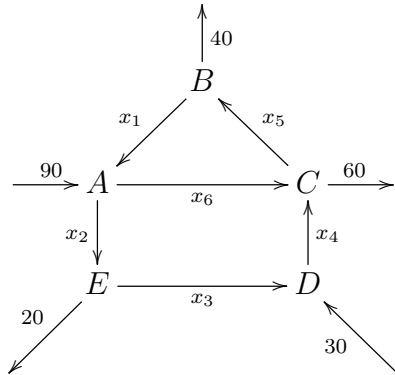
Solution: it has infinitely many solutions  $\iff \text{rank } A = \text{rank } (A | b) < 3 \iff e = -4$  and  $f = -6$ .

(iii) no solutions

(1)

Solution: it has no solutions  $\iff \text{rank } A < \text{rank } (A | b) \iff e = -4$  and  $f \neq -6$ .

6. Consider the network of streets with intersections  $A$ ,  $B$ ,  $C$ ,  $D$  and  $E$  below. The arrows indicate the direction of traffic flow along the one-way streets, and the numbers refer to the exact number of cars observed to enter or leave  $A$ ,  $B$ ,  $C$ ,  $D$  and  $E$  during one minute. Each  $x_i$  denotes the unknown number of cars which passes along the indicated streets during the same period.



6(a) Write down a system of linear equations which describes the traffic flow **together with all the constraints on the variables**  $x_i$ ,  $i = 1, \dots, 6$ . (1)

(Do not perform any operations on your equations: this is done for you in (b). Do not simply copy out the equations implicit in (b). You will not get any marks if you do this)

Intersection	Flow in = Flow out
A	$90 + x_1 = x_2 + x_6$
B	$x_5 = x_1 + 40$
C	$x_4 + x_6 = x_5 + 60$
D	$30 + x_3 = x_4$
E	$x_2 = x_3 + 20$

Oneway streets give constraints  $x_i \geq 0$ ,  $i = 1, \dots, 6$ . Since each  $x_i$  is a number of cars,  $x_i$  has to be an integer.

**6(b)** The reduced row-echelon form of the augmented matrix of the system in part (a) is

$$\left( \begin{array}{cccccc|c} 1 & 0 & 0 & 0 & -1 & 0 & -40 \\ 0 & 1 & 0 & 0 & -1 & 1 & 50 \\ 0 & 0 & 1 & 0 & -1 & 1 & 30 \\ 0 & 0 & 0 & 1 & -1 & 1 & 60 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

Give the general solution of this system (1)

(Ignore the constraints from (a) at this point)

Solution: Set  $x_5 = s$  and  $x_6 = t$  to be parameters. Then

$$\begin{aligned} x_1 &= -40 + s \\ x_2 &= 50 + s - t \\ x_3 &= 30 + s - t \\ x_4 &= 60 + s - t \\ x_5 &= s \\ x_6 &= t \end{aligned}$$

**6(c)** If the road  $ED$  was closed in the middle due to roadwork, find the minimum flow along the road  $AC$  **using your results from (b)** (2)

(you must justify all your answers: correct answer without justification is 1pt only)

Solution:  $ED$  is closed  $\iff x_3 = 0 \iff s - t = -30$ . Then assuming the constraints  $x_i \geq 0$  we obtain the system of inequalities

$$\left\{ \begin{array}{l} x_1 = -40 + s \geq 0 \iff s \geq 40 \\ x_2 = 50 - 30 = 20 \\ x_3 = 0 \\ x_4 = 60 - 30 = 30 \\ x_5 = s \geq 0 \\ x_6 = s - 30 \geq 0 \iff s \geq 30 \end{array} \right.$$

The flow along  $AC$  is  $x_6 = s + 30$  so that  $x_6 \geq 70$ . The minimum flow along  $x_6$  is 70.

7. Mark whether each of the following statements is TRUE or FALSE in the respective box.

(each correct answer is 1/4pt)

- For any three  $3 \times 3$  matrices  $A$ ,  $B$  and  $C$ , we have  $(AB)C = A(BC)$ .

ANSWER:  TRUE

- There exists a non-zero matrix  $A$  such that  $A^2$  is the zero-matrix.

ANSWER:  TRUE

- For any two  $3 \times 3$ -matrices  $A$  and  $B$ , we have  $A^2 - B^2 = (A - B)(A + B)$

ANSWER:  FALSE

- Multiplying a  $2 \times 3$ -matrix  $A$  by a  $3 \times 4$ -matrix  $B$  one gets  $2 \times 4$ -matrix  $AB$

ANSWER:  TRUE

8. If  $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$ , and  $B$  is a  $3 \times 5$  matrix,

then the second row of the matrix  $A \cdot B$  is (1)

cross (X) the correct answer:

- A the same as the first row of  $B$
- B the sum of the first and the second rows of  $B$
- C the sum of the first, the second and the third rows of  $B$
- D the sum of the first and the third rows of  $B$
- E the same as the third row of  $B$
- F the sum of the second and the third rows of  $B$

Solution: Write  $B = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}$  in block form, where  $r_i$  is the  $i$ -th row of  $B$ . Then

$$A \cdot B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} = \begin{pmatrix} r_1 + r_2 + r_3 \\ r_3 \\ r_1 + r_3 \end{pmatrix}$$

So the correct answer is  E.

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The last page (use it for computations)