Université d'Ottawa · University of Ottawa
Faculté des sciences
Mathématiques et de statistique
Faculty of Science
Mathematics and Statistics

MAT1320B — **Final Exam** — **8 December 2017**

Calculus I — **Elizabeth Maltais**

LAST NAME: ____________________

FIRST NAME: ____________________

STUDENT NUMBER: ____________________

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/3	/6	/3	/3	/6	/6	/4	/6	/6	/8	/4	/5	/+4	/60

[3] **1.** Find the derivative directly from the definition for the function $f(x) = \dfrac{1}{x+2}$. You must use the definition, not some other method.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{(x+h)+2} - \frac{1}{x+2}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{x+2}{(x+h+2)(x+2)} - \frac{(x+h+2)}{(x+h+2)(x+2)}}{h}$$

$$= \lim_{h\to 0} \frac{x+2-x-h-2}{(x+h+2)(x+2)(h)}$$

$$= \lim_{h\to 0} \frac{-\cancel{h}}{(x+h+2)(x+2)(\cancel{h})}$$

$$= \lim_{h\to 0} \frac{-1}{(x+h+2)(x+2)}$$

$$= \frac{-1}{(x+0+2)(x+2)}$$

$$= -\frac{1}{(x+2)^2}$$

[6] **2.** Find the derivative of each function.

a) $f(x) = \sin(\ln(x^2))$

$$f'(x) = \cos(\ln(x^2))\left(\frac{1}{x^2}\right)(2x)$$

b) $g(t) = t^{\ln(t)}$

$$\ln(g(t)) = \ln(t^{\ln(t)})$$

$$\Rightarrow \ln(g(t)) = \ln(t)\ln(t)$$

$$\Rightarrow \frac{1}{g(t)} \cdot g'(t) = \frac{1}{t}\ln(t) + \ln(t)\cdot\frac{1}{t}$$

$$\Rightarrow g'(t) = g(t)\left[\frac{1}{t}\ln(t) + \ln(t)\cdot\frac{1}{t}\right]$$

$$\Rightarrow g'(t) = \left(t^{\ln(t)}\right)\cdot\left[\frac{2\ln(t)}{t}\right]$$

[3] **3.** Estimate the value of $f(0.1)$ using a linearization of $f(x) = \tan(x) + 1$. Choose the point a of the linearization appropriately.

$$L(x) = f(a) + f'(a)(x-a)$$

Let $a = 0$.

$$f(x) = \tan(x) + 1 \qquad f(0) = \tan(0) + 1 = 0 + 1 = 1$$

$$f'(x) = \sec^2(x) \qquad f'(0) = \sec^2(0) = \frac{1}{(\cos(0))^2} = \frac{1}{1^2} = 1$$

$$\therefore L(x) = 1 + 1(x-0) = 1 + x$$

$$\therefore f(0.1) \approx L(0.1) = 1 + 0.1$$

[3] **4.** Show that $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\displaystyle\int_3^{x^2} \frac{1/2}{1+t}\,\mathrm{d}t + \int_{\tan^{-1}x}^{2} \tan(t)\,\mathrm{d}t\right)$ is zero.

By FTC 1,

$$\frac{d}{dx}\left(\int_3^{x^2} \frac{1/2}{1+t}dt + \int_{\tan^{-1}(x)}^{2} \tan(t)\,dt\right)$$

$$= \frac{1/2}{1+x^2}(x^2)' + -\frac{d}{dx}\int_2^{\tan^{-1}(x)} \tan(t)\,dt$$

$$= \frac{\frac{1}{2}(2x)}{1+x^2} - \left(\tan(\tan^{-1}(x)) \cdot (\tan^{-1}(x))'\right)$$

$$= \frac{x}{1+x^2} - \left(\underbrace{\tan(\tan^{-1}(x))}_{=x} \cdot \left(\frac{1}{1+x^2}\right)\right)$$

$$= \frac{x}{1+x^2} - \frac{x}{1+x^2}$$

$$= 0$$

[6] **5.** Consider the curve defined by $y + xe^y = x^2$.

a) Give $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y.

$$y + xe^y = x^2$$

$$\Rightarrow \frac{dy}{dx} + 1 \cdot e^y + x \cdot e^y \cdot \frac{dy}{dx} = 2x$$

$$\Rightarrow \frac{dy}{dx} + x \cdot e^y \cdot \frac{dy}{dx} = 2x - e^y$$

$$\Rightarrow \frac{dy}{dx}\left(1 + xe^y\right) = 2x - e^y$$

$$\Rightarrow \frac{dy}{dx} = \frac{2x - e^y}{1 + xe^y}$$

b) Find all values of x such that the point $(x, 0)$ is on the curve defined by the above equation. For each of these give the slope of the tangent line to the curve at that point.

$y + xe^y = x^2$ at $(x,0)$ $y=0$

$$\Rightarrow 0 + xe^0 = x^2$$

$$\Rightarrow x = x^2$$

$$\Rightarrow 0 = x^2 - x$$

$$\Rightarrow 0 = x(x-1)$$

$x=0$ $x=1$

at $(0,0)$, slope $= \dfrac{dy}{dx} = \dfrac{2(0) - e^0}{1 + (0)e^0} = -1$

at $(1,0)$, slope $= \dfrac{dy}{dx} = \dfrac{2(1) - e^0}{1 + 1e^0} = \dfrac{2-1}{2} = \dfrac{1}{2}$

[6] **6.** Find each of the limits.

a) $\lim_{x\to 3^+} \frac{1-e^{x-3}}{(x-3)^2}$ $\to 1-e^{3-3} \to 1-e^0 \to 0$ (indeterminate form $\frac{0}{0}$)

$\to (3-3)^2 \to 0$

$$\overset{\text{l'Hospital's}}{=} \lim_{x\to 3^+} \frac{(1-e^{x-3})'}{((x-3)^2)'}$$

$$= \lim_{x\to 3^+} \frac{-e^{x-3}}{2(x-3)} \quad \begin{matrix} \to -e^{3-3} \to -1 \\ \to 2(3^+-3) \to 0^+ \end{matrix} \quad \frac{-1}{0^+}$$

$$= -\infty$$

b) $\lim_{x\to 0}(x^2+1)^{1/x}$ $= \lim_{x\to 0} e^{\ln\left((x^2+1)^{(1/x)}\right)}$

$$= \lim_{x\to 0} e^{\frac{1}{x}\ln(1+x^2)}$$

$$= e^{\left(\lim_{x\to 0} \frac{1}{x}\ln(1+x^2)\right)}$$

this is indeterminate $\frac{0}{0}$

$$= e^{\left(\lim_{x\to 0} \frac{\ln(1+x^2)}{x}\right)}$$

$$\overset{\text{l'Hospital's}}{=} e^{\left(\lim_{x\to 0} \frac{\frac{2x}{1+x^2}}{1}\right)}$$

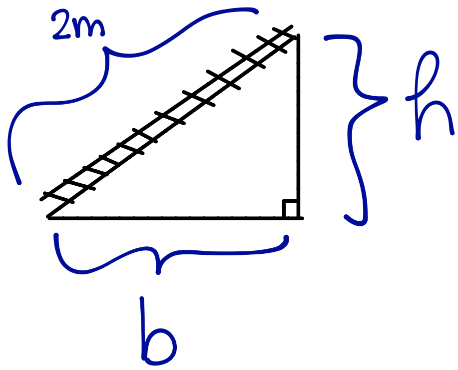
$$= e^{\left(\lim_{x\to 0} \frac{2x}{1+x^2}\right)}$$

$$= e^{\frac{2(0)}{1+0^2}} = e^0 = 1$$

← correction

[4] **7.** A ladder of length 2m is leaning against wall. The top of the ladder slides vertically down the wall while the bottom slides horizontally directly away from the wall.

When the bottom of the ladder is 1m from the wall and moving at 0.1m/s, how fast is the top of the ladder falling?



when $b = 1\text{m}$ and $\frac{db}{dt} = 0.1 \text{ m/s}$ what is $\frac{dh}{dt}$?

eq. $b^2 + h^2 = 2^2$ $\Longrightarrow$ $h = \sqrt{2^2 - b^2}$

when $b = 1$, $h = \sqrt{2^2 - 1^2} = \sqrt{3}$ m

$\Longrightarrow 2b \cdot \frac{db}{dt} + 2h \cdot \frac{dh}{dt} = 0$

$\Longrightarrow 2(1)(0.1) + 2(\sqrt{3}) \cdot \frac{dh}{dt}$

$\Longrightarrow \frac{dh}{dt} = \frac{-2(1)(0.1)}{2\sqrt{3}} = -\frac{0.1}{\sqrt{3}}$ m/s

$\therefore$ the top of the ladder is falling at a rate of $\frac{0.1}{\sqrt{3}}$ m/s.

[6] **8.** Evaluate each integral.

a) $\int xe^x \, dx$

parts $u = x$ $v' = e^x$
$u' = 1$ $v = e^x$

$\int uv' = uv - \int u'v$

$= xe^x - \int 1 \cdot e^x \, dx$

$= xe^x - \int e^x \, dx$

$= xe^x - e^x + C$

b) $\int (\ln(x))^2 \, dx$

parts $u = (\ln(x))^2$ $v' = 1$
$u' = 2(\ln(x))\left(\frac{1}{x}\right)$ $v = x$

$= (\ln(x))^2(x) - \int 2\ln(x) \cdot \left(\frac{1}{\cancel{x}}\right) \cdot (\cancel{x}) \, dx$

$= x(\ln(x))^2 - 2\int \ln(x) \, dx$

parts
$u = \ln(x)$ $v' = 1$
$u' = \frac{1}{x}$ $v = x$

$= x(\ln(x))^2 - 2\left[\ln(x) \cdot x - \int \left(\frac{1}{x}\right)(x) \, dx\right]$

$= x(\ln(x))^2 - 2\left(x\ln(x) - \int dx\right)$

$= x(\ln(x))^2 - 2\left(x\ln(x) - x\right) + C$

[6] **9.** Evaluate each integral.

a) $\displaystyle\int \frac{4}{(x+1)^2(x-1)}\,\mathrm{d}x$

partial fractions: $\dfrac{4}{(x+1)^2(x-1)} = \dfrac{A}{x+1} + \dfrac{B}{(x+1)^2} + \dfrac{C}{x-1}$

$$\Rightarrow \frac{4}{(x+1)^2(x-1)} = \frac{A(x+1)(x-1) + B(x-1) + C(x+1)^2}{(x+1)^2(x-1)}$$

$$\Rightarrow 4 = A(x^2-1) + B(x-1) + C(x^2+2x+1)$$

plug in $x=1 \Rightarrow 4 = 0A + 0B + 4C \Rightarrow C = 1$

plug in $x=-1 \Rightarrow 4 = 0A - 2B + 0C \Rightarrow B = -2$

also

$$\Rightarrow 4 = (A+C)x^2 + (B+2C)x + (-A-B+C)$$

$$\Rightarrow A+C=0 \qquad B+2C=0 \qquad -A-B+C=4$$

$$\therefore A = -C = -1$$

Thus $\displaystyle\int \frac{4}{(x+1)^2(x-1)}\,dx = \int\left(\frac{-1}{x+1} + \frac{-2}{(x+1)^2} + \frac{1}{x-1}\right)dx$

$$= -\int \frac{1}{x+1}\,dx - 2\int (x+1)^{-2}\,dx + \int \frac{1}{x+1}\,dx$$

$$= -\ln|x+1| - 2\left(\tfrac{1}{-1}(x+1)^{-1}\right) + \ln|x+1| + C$$

$$= -\ln|x+1| + 2(x+1)^{-1} + \ln|x+1| + C.$$

b) $\displaystyle\int \frac{x}{\sqrt{(x+2)^2-1}}\,\mathrm{d}x$ $\longrightarrow$ trig sub: $x+2=\sec\theta$

$\Rightarrow x=\sec\theta-2$

$\Rightarrow \dfrac{dx}{d\theta}=\sec\theta\tan\theta-0$

$\Rightarrow dx=\sec\theta\tan\theta\,d\theta$

$$=\int\frac{(\sec\theta-2)}{\sqrt{\sec^2\theta-1}}(\sec\theta\tan\theta\,d\theta)$$

$$=\int\frac{(\sec\theta-2)(\sec\theta\tan\theta)}{\sqrt{\tan^2\theta}}\,d\theta$$

$$=\int\frac{(\sec\theta-2)(\sec\theta\cancel{\tan\theta})}{\cancel{\tan\theta}}\,d\theta$$

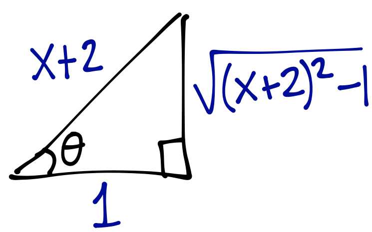
$$=\int(\sec\theta-2)(\sec\theta)\,d\theta$$

$$=\int\sec^2\theta-2\sec\theta\,d\theta$$

$$=\int\sec^2\theta-2\int\sec\theta$$

$$=\tan\theta-2\ln|\sec\theta+\tan\theta|+C$$

Since $x+2=\sec\theta$



$$=\left(\frac{\sqrt{(x+2)^2-1}}{1}\right)-2\ln\left|x+2+\frac{\sqrt{(x+2)^2-1}}{1}\right|+C$$

[8] **10.** Evaluate each integral.

a) $\displaystyle\int_0^1 \frac{e^x}{e^{2x}+5e^x+6}\,dx$

Sub:

$u = e^x$

$\frac{du}{dx} = e^x \Rightarrow dx = \frac{du}{e^x}$

$x=1 \Rightarrow u = e^1$

$x=0 \Rightarrow u=e^0=1$

$$= \int_{u=1}^{u=e} \frac{\cancel{e^x}}{u^2+5u+6}\cdot\frac{du}{\cancel{e^x}}$$

$$= \int_1^e \frac{1}{u^2+5u+6}\,du$$

$$= \int_1^e \frac{1}{(u+2)(u+3)}\,du$$

partial fractions

$$\frac{1}{(u+2)(u+3)} = \frac{A}{u+2} + \frac{B}{u+3}$$

$\Rightarrow 1 = A(u+3)+B(u+2)$

plug in $u=-3 \Rightarrow 1 = 0A - B \Rightarrow B=-1$

plug in $u=-2 \Rightarrow 1 = A+0B \Rightarrow A=1$

$$= \int_1^e \left(\frac{1}{u+2} + \frac{-1}{u+3}\right)du$$

$$= \Big[\ln|u+2| - \ln|u+3|\Big]_1^e$$

$$= \left(\ln|e+2| - \ln|e+3|\right) - \left(\ln|1+2| - \ln|1+3|\right)$$

$$= \ln(e+2) - \ln(e+3) - \ln(3) + \ln(4).$$

b) $\displaystyle\int_0^{\pi/4} \frac{e^{\tan x}}{\cos^2 x}\,\mathrm{d}x$

$u = \tan(x)$

$\frac{du}{dx} = \sec^2 x \Rightarrow dx = \frac{du}{\sec^2 x}$

$x = \pi/4 \Rightarrow u = \tan(\frac{\pi}{4}) = 1$

$x = 0 \Rightarrow u = \tan(0) = 0$

$$= \int_{u=0}^{u=1} \frac{e^u}{\cos^2(x)} \cdot \frac{du}{\sec^2(x)}$$

$$= \int_0^1 \frac{e^u}{\cancel{\cos^2}(x)\left(\frac{1}{\cancel{\cos^2 x}}\right)} du$$

$$= \int_0^1 e^u du$$

$$= e^u \Big|_0^1$$

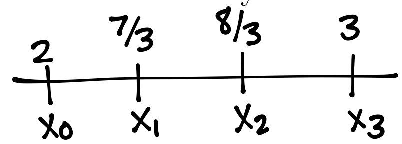
$$= e^1 - e^0$$

$$= e - 1$$

[4] **11.** We wish to evaluate $\int_2^3 \ln(x)\,\mathrm{d}x$ numerically.

a) Give an expression for the Riemann sum for $\int_2^3 \ln(x)\,\mathrm{d}x$ using $n = 3$ rectangles and the right-hand rule. You do not need to evaluate your expression numerically.

$f(x) = \frac{1}{x}$ interval $[2,3]$ $n = 3$ $\Delta x = \frac{3-2}{3} = \frac{1}{3}$



$$R_3 = \sum_{i=1}^{3} f(x_i)\Delta x = \left(f(7/3) + f(8/3) + f(3)\right)\left(\frac{1}{3}\right)$$

$$\therefore\ R_3 = \left(\ln\left(\frac{7}{3}\right) + \ln(8/3) + \ln(3)\right)\cdot\left(\frac{1}{3}\right)$$

b) The difference between $\int_a^b f(x)\,\mathrm{d}x$ and the approximation using Simpson's method with n subintervals (or "rectangles") is at most $\frac{K(b-a)^5}{180n^4}$, where $|f''''(x)| \le K$ on $[a,b]$. Give an expression for the value of n required so that Simpsons's method applied to $\int_2^3 \ln(x)\,\mathrm{d}x$ is accurate to within 0.00001. You do not need to compute Simpson's method, nor evaluate your expression numerically. An expression for n suffices.

$f(x) = \ln(x) \Rightarrow f'(x) = \frac{1}{x} = x^{-1} \Rightarrow f''(x) = -x^{-2} \Rightarrow f'''(x) = 2x^{-3} \Rightarrow f^{(4)}(x) = \frac{-6}{x^4}$

on $[2,3]$ $|f^{(4)}(x)| = \left|\frac{-6}{x^4}\right| = \frac{6}{x^4} \le \frac{6}{2^4}$ since $\frac{6}{x^4}$ is decreasing on $[2,3]$

correction

$\Rightarrow$ we can use $K = \frac{6}{16}$ in the error bound.

We want n such that $\frac{K(b-a)^5}{180n^4} \le 0.00001$

Corrections!

$$\Rightarrow \frac{\frac{6}{16}(3-2)^5}{180n^4} \le 0.00001$$

$$\Rightarrow n^4 \ge \frac{\frac{6}{16}}{180(0.00001)} \Rightarrow n \ge \sqrt[4]{\frac{6/16}{180(0.00001)}}$$ (and n should be even).

[5] **12.** Consider the following function, and its derivatives.

$$f(x) = \frac{e^{-x}}{x^2}$$
$$f'(x) = \frac{-e^{-x}(x+2)}{x^3}$$
$$f''(x) = \frac{e^{-x}(x^2+4x+6)}{x^4}$$

a) Identify all horizontal and vertical asymptotes. (of f)

$\boxed{\text{VA when } x=0}$

$$\lim_{x\to 0^-} \frac{e^{-x}}{x^2} \to \frac{1}{0^+} = \infty \qquad \lim_{x\to 0^+} \frac{e^{-x}}{x^2} \to \frac{1}{0^+} = \infty$$

HA? $\lim_{x\to\infty} \frac{e^{-x}}{x^2} \begin{matrix}\to 0\\ \to \infty\end{matrix} = 0$ $\boxed{\text{HA } y=0 \text{ as } x\to\infty}$

$$\lim_{x\to-\infty} \underbrace{\frac{e^{-x}}{x^2}}_{\frac{\infty}{\infty}} \overset{\text{Hosp.}}{=} \lim_{x\to-\infty} \underbrace{\frac{-e^{-x}}{2x}}_{\frac{-\infty}{-\infty}} \overset{\text{Hosp.}}{=} \lim_{x\to-\infty} \underbrace{\frac{e^{-x}}{2}}_{\frac{\infty}{2}} = \infty$$

b) Determine where it is increasing and where it is decreasing. Identify all extrema (local maximum and minimum).

crit. #s $0 = f'(x) \Rightarrow 0 = \frac{-e^{-x}(x+2)}{x^3} \Rightarrow 0 = -e^{-x}(x+2)$

$-e^{-x}$: never zero; $(x+2)$: $x=-2$

intervals of domain	$(-\infty,-2)$	-2	$(-2,0)$	0	$(0,\infty)$
sign of $f'(x)$	$-$		$+$		$-$
behaviour	decreasing on $(-\infty,-2)$		increasing on $(-2,0)$		decreasing on $(0,\infty)$

local minimum at $(-2, f(-2)) = (-2, \frac{e^2}{4})$

c) Determine where it is concave up and where it is concave down. Identify all inflection points.

IP candidates: $0 = f''(x) \Rightarrow 0 = \frac{e^{-x}(x^2+4x+6)}{x^4} \Rightarrow 0 = e^{-x}(x^2+4x+6)$

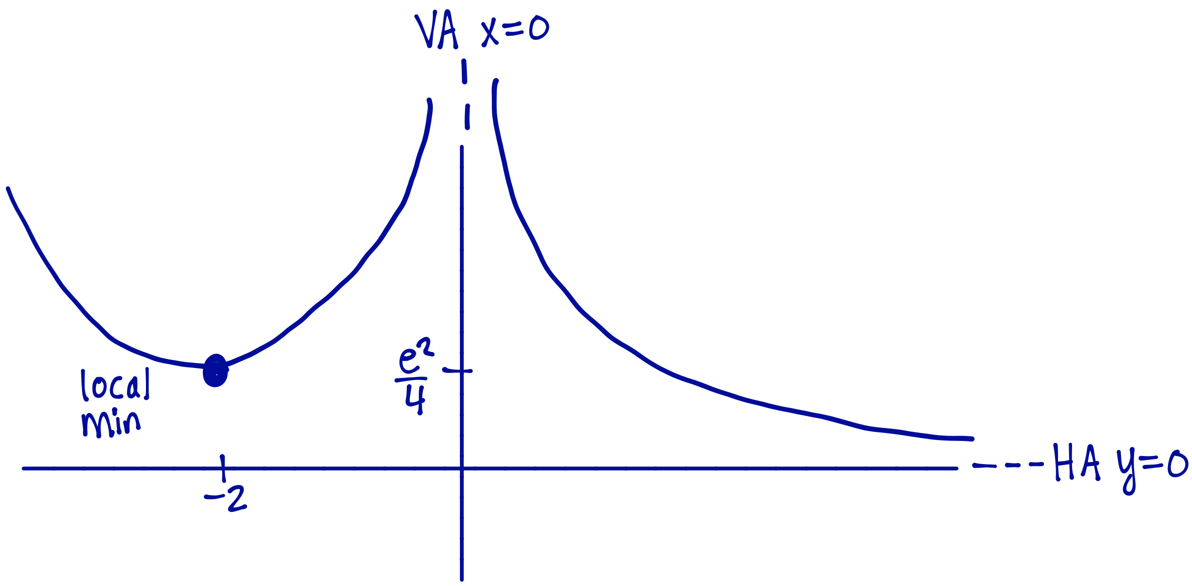
e^{-x}: never zero; (x^2+4x+6): $x = \frac{-4 \pm \sqrt{16-4(1)(6)}}{2(1)}$

no solutions x^2+4x+6 is an irreducible quadratic

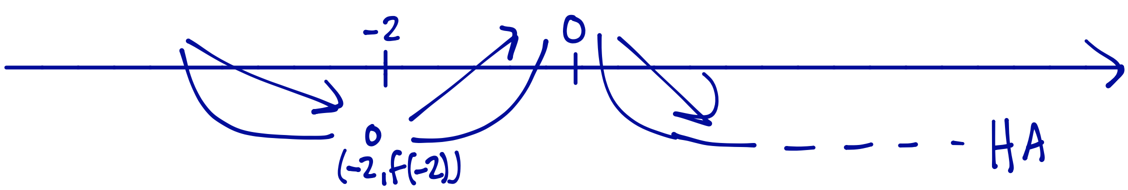
intervals of domain	$(-\infty,0)$	0	$(0,\infty)$
sign of $f''(x)$	$+$		$+$
behaviour	CONC. UP		CONC. UP

No inflection points.

d) Sketch the function, labelling the extrema, inflection points, asymptotes and the intercepts.



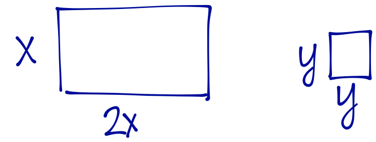
mini-map:



$f(x) = \frac{e^{-x}}{x^2}$ has no x-intercepts and is always positive

[+4] **13.** (bonus) Consider a rectangle of dimensions $2x \times x$ and a square of dimensions $y \times y$.

If the sum of the perimeters of the rectangle and the square is ℓ, find the value of x and y (in terms of ℓ) that minimize the sum of the areas of the rectangle and the square.



perimeter of rectangle + perimeter of square = ℓ

$$\Rightarrow \quad 2x+x+2x+x \; + y+y+y+y = \ell$$

$$\Rightarrow \quad 6x+4y=\ell$$

Area of rectangle + area of square $= (2x)\times(x) + (y)\times(y)$

Sum of Areas $= 2x^2+y^2$ ← to minimize

Since $6x+4y=\ell$, $\quad y = \dfrac{\ell-6x}{4} \Rightarrow$ sum of areas $= A = 2x^2 + \left(\dfrac{\ell-6x}{4}\right)^2$

To maximize: $A(x) = 2x^2 + \left(\dfrac{\ell-6x}{4}\right)^2$

$$A'(x) = 4x + 2\left(\frac{\ell-6x}{4}\right)\left(-\frac{6}{4}\right) = 4x - 3\left(\frac{\ell}{4}-\frac{3x}{2}\right) = 4x - \frac{3\ell}{4} + \frac{9}{2}x$$

solve

$$0 = A'(x) \Rightarrow 0 = \frac{17}{2}x - \frac{3\ell}{4} \Rightarrow x = \frac{6\ell}{4(17)} = \frac{3\ell}{34}$$

intervals of A: 0 $\left(0, \frac{3\ell}{34}\right)$ $\frac{3\ell}{34}$ $\left(\frac{3\ell}{34}, \infty\right)$

Sign of A': $A'\left(\frac{2\ell}{34}\right) = \frac{17}{2}\left(\frac{2\ell}{34}\right) - \frac{3\ell}{4} = \frac{\ell}{2} - \frac{3\ell}{4} < 0$ $\quad$ $A'(\ell) = \frac{17}{2}\ell - \frac{3\ell}{4} > 0$

$-$ $\quad$ $+$

behaviour: decreasing $\quad$ increasing

Since $A(x)$ decreases for $0 < x < \frac{3\ell}{34}$, then increases for all $x > \frac{3\ell}{34}$, $A(x)$ attains an absolute minimum when $x = \frac{3\ell}{34}$ $\therefore$ $y = \dfrac{\ell - 6\left(\frac{3\ell}{34}\right)}{4}$