1. Proofs, Sets, and Functions

One purpose of this teaching material is to help improve the students' ability to understand and create proofs. This is done in Section 1.1. In Section 1.2 we consider sets, including arbitrary collections of sets. Sections 1.3 and 1.4 deal with functions and mathematical induction.

1.1 Proofs. Many of the statements we prove in this section are most probably known to the reader. It is the technique of proof we wish to emphasize.

Typical mathematical statements that require proof are the so called *a conditional statements* or we call them also *an implications*:

$$\text{if} \quad p \quad \text{holds,} \quad \text{then} \quad q \quad \text{holds} \quad [\text{or,} \quad p \quad \text{implies} \quad q] \quad [\text{or,} \quad p \Rightarrow q] \qquad (1)$$

In the above, p and q are statements; p is called *the hypothesis* and q is called *the conclusion*.

Direct proofs. The direct method to prove the conditional statement (1): We assume that p holds (is true) and we want to show that q holds also. In other words, by the truth of p we deduce the truth of q; or in other words, p implies q. Here is an example that illustrate the direct method to do proofs.

Proposition 1.1. If n is an even integer, then n^2 is an even integer.

Proof. The hypothesis p is the statement: *n is an even integer* and the conclusion q is the statement: n^2 is an even integer.

So, we assume that n is an even integer hence, $n = 2k$ for some integer k. Then in view of this, $n^2 = 4k^2 = 2\,(2k^2)$, where $2k^2$ is an integer therefore, n^2 is an even integer.

Remark. In Proposition 1.1, the hypothesis p is the statement: *n is an even integer* and the conclusion q is the statement: n^2 is an even integer. In order to prove something first, we have to know and understand well the definitions (the notions) of the terms in the proposition. In Proposition 1.1 we need to know what is an even integer. Also, in the proof, note that we have used the existential quantifier "for some" that is equivalent to "there exists", symbolically denoted by $\exists$: If n is an integer then, there exists (symbolically $\exists$) an integer k such that $n = 2k$.

The other basic quantifier is "for all" equivalently "for every", "for each", "for any", denoted symbolically by $\forall$.

At the very start of the proof, we should draw a strategy of the proof asking ourselves: **WHAT IS THAT I MUST DO IN ORDER TO SHOW THAT** n^2 is an even integer, given that n is an even integer. Basically you asked yourself: What are the steps that I am to follow till the final conclusion that is typically being "*therefore, q holds*" that, in the case of Proposition 1.1 is: *therefore, n^2 is integer.*

The converse statement of a conditional statement. The converse of the conditional statement:

$$(\text{if } p \text{ holds, then } q \text{ holds}) \quad \textit{is} \quad (\text{if } q \text{ holds, then } p \text{ holds})\,.$$

The converse statement of one conditional statement may or may not be true. For example, the converse of the conditional statement in Proposition 1.1 is true: *If the integer n^2 is even, then n is also even.*
However, the converse of the basic calculus conditional statement: *If a function f is differentiable at a point a, the f is continuous at the point a* is false (not true).

Biconditional statement. Given the statements p and q, the biconditional statement

$$p \quad \text{if and only if} \quad q \qquad \text{also denoted by} \qquad p \Leftrightarrow q \qquad \text{or by} \qquad p \quad \text{iff} \quad q$$

means that the both conditional statement are true (hold)

$$\text{if } p, \text{ then } q \qquad \text{and} \qquad \text{if } q, \text{ then } p\,.$$

Hence, in order to prove the biconditional statement $p \Leftrightarrow q$, we must prove both conditional statements: $p \Rightarrow q$ and $q \Rightarrow p$.

We shall denote by $\mathbf{R}$ the set of all real numbers and by $\mathbf{N}$ the set of all positive integers $\{1, 2, 3, \cdots\}$ and $<$ denotes the usual ordering in $\mathbf{R}$.

Definition 1.1 Let A be a subset of $\mathbf{R}$. The set A is unbounded above if for each positive real number x there exists a number a in A such that $x < a$.
Symbolically, A is unbounded above if $\forall\ x > 0$, $\exists\ a$ in A such that $x < a$. Note that the choice of a depends on x.

Remark. Definitions are to be interpreted in the "if and only if" sense, even though it is common practice not to state them this way. For example, in Definition 1.1 the "if" is actually "if and only if".

Archimedean Principle. If x is a positive real number, then there exists a positive integer n such that $1/n < x$. Symbolically, $\forall\ x > 0$, $\exists\ n$ in $\mathbf{N}$ such that $1/n < x$. Note that n depends on x.

Proposition 1.2 **N** is unbounded above if and only if the Archimedean Principle holds.

(Direct) Proof. **(a)** Assume that **N** is unbounded above. We need to show that the Archimedean Principle holds. Let $x > 0$. (The reader should now ask: How to continue the proof?) We have to show that there is a positive integer n such that $1/n < x$. Because $x > 0$, then $1/x > 0$. Because we assume that **N** is unbounded above then there is a positive integer n such that $1/x < n$ therefore, $1/n < x$. Hence, for each positive x there exists a positive integer n such that $1/n < x$ that is the Archimedean Principle. Hence, if **N** is unbounded above, then the Archimedean Principle holds.
(b) Conversely, suppose that the Archimedean Principle holds. Then, we have to show that **N** is unbounded above. Let x be a positive number. Then $1/x$ is also positive and by the Archimedean Principle there is a positive integer n such that $1/n < 1/x$ and thus $x < n$. Therefore, **N** is unbounded above.

Remark. Note that we have not shown that **N** is unbounded above, nor have we shown that the Archimedean Principle is true.
We have shown that [**N** is unbounded above] *is a true statement* **if and only if** [the Archimedean Principle] *is a true statement, or that the statement* [**N** is unbounded above] *is logically equivalent to* [the Archimedean Principle].

Indirect Proofs. There are two types of indirect proofs: **the contrapositive argument and the contradiction argument**.

Proofs by contrapositive argument. The contrapositive (the negation) of the conditional statement "if p, then q" is the statement "if not q, then not p". Obviously, a conditional statement and its contrapositive have the same truth value, i.e., they are equivalent. Thus, to prove a conditional statement one can prove its contrapositive. Since we shall negate many statements in our considerations, let us give a basic rule for a negation of a statement: Change all universal quantifiers to existential quantifiers; change all existential quantifiers to universal quantifiers; and negate the main clause.

Proposition 1.3 Let n be an integer. If n^2 is an odd integer, then n is an odd integer.

Proof. This is the contrapositive of Proposition 1.1.

Proofs by contradiction (by assuming to the contrary). To prove the conditional statement "if p, then q" by contradiction, one assumes that p is true and q is false and *hunts* for a contradiction. Once a contradiction is reached, it follows that if p is true, then q must also be true. Sometimes the contradiction is easy to be reached and it is clear, but sometimes it is unclear and difficult to be reached.

In order to illustrate proofs by contradiction in the next two propositions we assume the usual order properties on $\mathbf{R}$ and: if a is a real number then $-a$ is the additive inverse of a,i.e., $-a + a = 0$.

Proposition 1.4 Let $a \in \mathbf{R}$. If $a > 0$, then $-a < 0$.

Proof. Assume $a > 0$ and that the statement $-a < 0$ is false. Then, $-a \geq 0$ hence, $-a + a > 0$ but $-a + a = 0$ so, $0 > 0$, which is an obvious contradiction. Therefore, $-a < 0$.

Proposition 1.5 Let $a \in \mathbf{R}$. If $a < \varepsilon$ for all $\varepsilon > 0$, then $a \leq 0$.

Proof 1. Assume to the contrary: $a < \varepsilon$ for all $\varepsilon > 0$ and $a > 0$. According to the Archimedean Principle there is a positive integer n such that $1/n < a$. However, with $\varepsilon = 1/n > 0$ we must have also that $a < 1/n$. Hence $1/n < a < 1/n$ so, $1/n < 1/n$ that is an obvious contradiction. Therefore, $a \leq 0$.

Proof 2. Assume to the contrary: $a < \varepsilon$ for all $\varepsilon > 0$ and $a > 0$. Then, with $\varepsilon = a/2 > 0$ we must have that $a < a/2$ that is an obvious contradiction. Therefore, $a \leq 0$.

Rational and irrational numbers. Prime numbers. A rational number is a real number that can be expressed in the form m/n , where m and n are integers and $n \neq 0$. An irrational number is a real number that is not a rational number. A prime number or simply a prime is a positive integer greater than 1 whose positive divisors are itself and 1. By divisors we mean integers that divide a given number exactly that is with zero remainder. For example $2, 3, 5, 7, 11, \ldots$ are prime numbers, whereas 9 is not a prime number because 3 is a divisor of 9. Also, if a prime divides a product of two integers, then this prime must divide at least one of these two integers.

Theorem 1.1 The number $\sqrt{2}$ is an irrational number.

Proof. Assume to the contrary, that $\sqrt{2}$ is a rational number. Since $\sqrt{2}$ is positive, there exist positive integers m and n such that $\sqrt{2} = m/n$ and m/n is in lowest terms. Note that we can always reduce a fraction m/n to its lowest terms. Then $\sqrt{2}n = m$ and $2n^2 = m^2$. The prime 2 divides $m^2 = m.m$ then 2 must divide m hence, m is even so, there is a positive integer k such that $m = 2k$. Then, $2n^2 = 4k^2$ and $n^2 = 2k^2$ hence, 2 divides $n^2 = n.n$ so, 2 divides n. Thus m/n is not in lowest term, which is a contradiction. Therefore, $\sqrt{2}$ is not a rational number hence, it is irrational number.

Prime divisors. Each positive integer greater than 1 is divisible by a prime, i.e., has a prime divisor.

Theorem 2.1 There are infinitely many primes.

Proof. Suppose to the contrary that there are only finitely many primes, say $p_1, p_2, \ldots, p_n$. Let

$$M = p_1 p_2 \cdots p_n + 1 \,.$$

Then M is an integer greater than 1 hence, it has a prime divisor. Thus some p_i divides M but also p_i divides $p_1 p_2 \cdots p_n$. In view os this, p_i must divide 1, which is a contradiction. Therefore, there are infinitely many primes.

Summing up, in order to prove a theorem:
(1) First, we have to know what the terms in the theorem mean.
(2) We need to have knowledge, i.e., to know with understanding facts - axioms, propositions, theorems on which we base the proof.
(3) We must know the end of the proof, and keeping the end in mind helps to prevent the line of reasoning from straying off course.

1.2 Sets. In this teaching material we emphasize on the techniques of proofs. We adopt the view-point of *naive* set theory considering the notion of a set as already known.

Basic Results and set operations. A set is well-defined collection of objects. By *well-defined* we mean that given a set and an object, it is possible to determine wether the object is or is not in this set. Each object of a set is called *an element* of the set (*a point* of the set) (*a member* of the set).
If A is a set and x is a point, then

$$x \in A \quad \text{denotes that} \quad x \quad \text{is an element of} \quad A$$

$$x \notin A \quad \text{denotes that} \quad x \quad \text{is not an element of} \quad A$$

A set can be defined either by listing the elements or by stating a property of its elements. For example

$$A = \{1, 3\} = \{x \in \mathbf{R} \,:\, x^2 - 4x + 3 = 0\}$$

where $\mathbf{R}$ denotes the set of real numbers. The *empty set* denoted by $\emptyset$ is the set with no elements. Thus

$$\emptyset = \{x \,:\, x^2 < 0\} = \{x \,:\, x \neq x\} = \{x \,:\, \frac{x}{x+1} = 1\}$$

and so on.

Definition 1.2 Let A and B be sets.
(1) A is a subset of B, denoted by $A \subset B$ or $B \supset A$, if for each x in A, x is in B that is: for $\forall x \in A, x \in B$.
(2) A is equal to B, denoted by $A = B$, if $A \subset B$ and $B \subset A$.
(3) A is a proper subset of B if $A \subset B$ and $A \neq B$.
Thus to prove that two sets are equal one must show that each of these two sets is a subset of the other. Also, because $\emptyset$ does not have elements, then $\emptyset \subset A$ for each set A.

Definition 1.3 Let A and B be sets.
(1) The *union* of the sets A and B denoted by $A \cup B$ is defined as

$$A \cup B = \{x \,:\, x \in A \quad \text{or} \quad x \in B\}.$$

The word "or" in the above definition is used in the *inclusive sense*, so that points that belong to both A and B also belong to the union $A \cup B$.
(2) The *intersection* of A and B denote by $A \cap B$ is defined as

$$A \cap B = \{x \,:\, x \in A \quad \text{and} \quad x \in B\}.$$

Thus $A \cap B \subset A \cup B$.
(3) The sets A and B are *disjoint* if $A \cap B = \emptyset$.

Proposition 1.6 Let A, B, and C be sets. Then
1. $A \cup A = A$ and $A \cap A = A$;
2. $A \cup \emptyset = A$ and $A \cap \emptyset = \emptyset$;
3. $A \subset A \cup B$; $B \subset A \cup B$ and $A \cap B \subset A$; $A \cap B \subset B$;
4. $A \cup B = B \cup A$ and $A \cap B = B \cap A$ (commutative property);
5. $A \cup (B \cup C) = (A \cup B) \cup C$ and $A \cap (B \cap C) = (A \cap B) \cap C$ (associative property)
6. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (distributive property)
7. $A \subset B$ if and only if $A \cup B = B$ and $A \subset B$ if and only if $A \cap B = A$.

Proof of the first equality in part 6. We have to show that

$$A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C) \qquad (2)$$

and

$$(A \cup B) \cap (A \cup C) \subset A \cup (B \cap C) \qquad (3).$$

To show (2), let $x \in A \cup (B \cap C)$. Then $x \in A$ or $x \in B \cap C$. If $x \in A$, then $x \in A \cup B$ and $x \in A \cup C$ hence, by the definition of intersection $x \in (A \cup B) \cap (A \cup C)$. If $x \in B \cap C$ then $x \in B$ and $x \in C$ and from here, $x \in A \cup B$ and $x \in A \cup C$ hence, by the definition of intersection $x \in (A \cup B) \cap (A \cup C)$.
To show (3), let $x \in (A \cup B) \cap (A \cup C)$. Then $x \in A \cup B$ and $x \in A \cup C$. If $x \in A$, then $x \in A \cup (B \cap C)$. Let $x \notin A$, then $x \in B$ and $x \in C$ and by the definition of intersection $x \in (B \cap C)$ and $x \in A \cup (B \cap C)$.
Proof of the first equality in part (7). We need to show that:

$$\text{if } A \subset B, \text{ then } A \cup B = B \qquad (4)$$

and

$$\text{if } A \cup B = B, \text{ then } A \subset B \qquad (5).$$

We consider (4). Let $A \subset B$. Then we have to show that $A \cup B \subset B$ and $B \subset A \cup B$. Suppose $x \in A \cup B$ that is $x \in A$ or $x \in B$. Suppose $x \in A$, then $x \in B$. Hence, $A \cup B \subset B$. Also, $B \subset A \cup B$.
We consider (5). Let $A \cup B = B$, we have to show that $A \subset B$. Suppose to the contrary, that A is not a subset of B. Then, there is $x \in A$ such that $x \notin B$. Then $x \in A \cup B$ but $x \notin B$ therefore, $A \cup B \neq B$ and we get a contradiction. Hence, $A \subset B$.

Definition 1.4 Let A and B be sets. The complement of B relative to A, denoted by $A \backslash B$ is defined as

$$A \backslash B = \{x \in A : x \notin B\}.$$

For example, if $\mathbf{R}$ denotes the set of real numbers and $\mathbf{Q}$ denotes the set of rational numbers that is

$$\mathbf{Q} = \left\{ \frac{m}{n} \ : \ m \text{ and } n \text{ intergers and } \ n \neq 0 \right\}$$

then $\mathbf{R}\backslash\mathbf{Q}$ is the set of irrational numbers. Also, $\mathbf{Q}\backslash\mathbf{R} = \emptyset$. In general, if $A \subset B$, then $A\backslash B = \emptyset$.

Proposition 1.7 Let A, B and C be sets. Then, 1. $A\backslash\emptyset = A$ and $A\backslash A = \emptyset$

2. If $A \subset B$, then $A\backslash B = \emptyset$.

3. De Morgan's laws:

$$A\backslash(B \cap C) = (A\backslash B) \cup (A\backslash C)$$
$$A\backslash(B \cup C) = (A\backslash B) \cap (A\backslash C)\,.$$

Proof. We prove the first equality in 3 leaving the rest as an exercise.We need to show that

$$A\backslash(B \cap C) \subset (A\backslash B) \cup (A\backslash C) \ \ (6) \quad \text{and} \quad (A\backslash B) \cup (A\backslash C) \subset A\backslash(B \cap C) \ \ (7)$$

To prove (6), suppose that $x \in A\backslash(B \cap C)$. Then, $x \in A$ but $x \notin B \cap C$ that is: either ($x \in B$ but $x \notin C$) or ($x \notin B$ but $x \in C$) or ($x \notin C$ and $x \notin B$). If ($x \in B$ but $x \notin C$ then $x \in A\backslash C$ and $x \in (A\backslash B) \cup (A\backslash C)$. If $x \notin B$ but $x \in C$, then $x \in A\backslash B$ and $x \in A\backslash B) \cup (A\backslash C)$. If $x \notin C$ and $x \notin B$, then $x \in A\backslash B$ and $x \in A\backslash C$ hence, $x \in A\backslash B) \cup (A\backslash C)$.

To prove (7), suppose that $x \in (A\backslash B) \cup (A\backslash C)$. Then, either $x \in A\backslash B$ or $x \in A\backslash C$. Suppose, $x \in A\backslash B$. Then, $x \in A$ but $x \notin B$ and from here, $x \notin B \cap C$ hence, $x \in A\backslash(B \cap C)$. Suppose $x \in A\backslash C$. Then, $x \in A$ but $x \notin C$ and from here, $x \notin B \cap C$ hence, $x \in A\backslash(B \cap C)$.

Arbitrary unions and intersections. We generalize Definition 1.3 to an arbitrary collection of sets. For example:

$$\{A_n \ : \ n \in \mathbf{N}\} \text{ where } A_n = (0, n) \text{ for each } n \in \mathbf{N}.$$

We can draw these sets on the real line.

Definition 1.5 Let $\mathcal{U}$ be a collection of sets.

1. The union of $\mathcal{U}$, denoted by $\bigcup\mathcal{U}$ is defined as

$$\bigcup\mathcal{U} = \{x \ : \ x \in A \ \text{ for at leat one } A \in \mathcal{U}\}\ .$$

2. If $\mathcal{U}$ is non-empty, the intersection of $\mathcal{U}$. denoted by $\bigcap\mathcal{U}$ is defined as

$$\bigcap\mathcal{U} = \{x \ : \ x \in A \ \text{ for all } A \in \mathcal{U}\}\ .$$

The above definition extend the notions of union and intersection given previously for $\mathcal{U} = \{A, B\}$, where A and B are sets: $\bigcup\mathcal{U} = A \cup B$ and $\bigcap\mathcal{U} = A \cap B$.
If $\mathcal{U}$ is an empty collection of sets then $\bigcup\mathcal{U} = \emptyset$ and we do not define $\bigcap\mathcal{U}$.

Equivalent formulation Definition 1.5 can be given in terms of an index set. Let I be a set, called the *index set*. Suppose A_α is a set for each $\alpha \in I$. Then $\mathcal{U} = \{A_\alpha : \alpha \in I\}$ and

$$\bigcup\mathcal{U} = \bigcup\{A_\alpha : \alpha \in I\} = \bigcup_{\alpha\in I} A_\alpha = \{x : x \in A_\alpha \text{ for some } \alpha \in I\}$$

$$\bigcap\mathcal{U} = \bigcap\{A_\alpha : \alpha \in I\} = \bigcap_{\alpha\in I} A_\alpha = \{x : x \in A_\alpha \text{ for all } \alpha \in I\}$$

If $I = \{1, 2, \ldots, n\}$ for some $n \in \mathbf{N}$, then we write

$$\bigcup_{i\in I} A_i = \bigcup_{i=1}^{n} A_i \quad \text{and} \quad \bigcap_{i\in I} A_i = \bigcap_{i=1}^{n} A_i\,.$$

If $I = \mathbf{N}$ where $\mathbf{N}$ is the index set of all positive integers then, we write

$$\bigcup_{i\in\mathbf{N}} A_i = \bigcup_{i=1}^{\infty} A_i \quad \text{and} \quad \bigcap_{i\in\mathbf{N}} A_i = \bigcap_{i=1}^{\infty} A_i\,.$$

Example 1.1

$$\bigcup_{i=1}^{n}(0, i) = (0, n); \quad \bigcap_{i=1}^{n}(0, i) = (0, 1)\,.$$

Example 1.2

$$\bigcup_{n=1}^{\infty}(0, n) = (0, \infty); \quad \bigcap_{n=1}^{\infty}(0, n) = (0, 1)\,.$$

Example 1.3

$$\bigcup_{n=1}^{\infty}(-n, n) = (-\infty, \infty) = \mathbf{R}; \quad \bigcap_{n=1}^{\infty}(-n, n) = (-1, 1)\,.$$

Example 1.4

$$\bigcup_{n=1}^{\infty}\{n\} = \mathbf{N}; \quad \bigcap_{n=1}^{\infty}\{n\} = \emptyset\,.$$

Example 1.5

$$\bigcup_{n=1}^{\infty}(0,1/n)=(0,1); \quad \bigcap_{n=1}^{\infty}(0,1/n)=\emptyset\,.$$

Let us prove that

$$\bigcap_{n=1}^{\infty}(0,1/n)=\emptyset\,.$$

Proof. Suppose to the contrary that $\bigcap_{n=1}^{\infty}(0,1/n)\neq\emptyset$. Then, these is a number x such that $x\in(0,1/n)$ for all positive integer n. Because $x>0$, by the Archimedean Principle, these is a positive integer n_0 such that $1/n_0<x$ hence, $x\notin(0,1/n_0)$ and we arrive at a contradiction. Therefore, $\bigcap_{n=1}^{\infty}(0,1/n)=\emptyset$.

Proposition 1.8 (De Morgan's laws) Let X be a set. Let I be a nonempty index set and A_α be a set for each $\alpha\in I$. Then

$$X\setminus\bigcup_{\alpha\in I}A_\alpha=\bigcap_{\alpha\in I}(X\setminus A_\alpha) \text{ and } X\setminus\bigcap_{\alpha\in I}A_\alpha=\bigcup_{\alpha\in I}(X\setminus A_\alpha)\,.$$

Proof. We prove the first equality. We have to show that

$$X\setminus\bigcup_{\alpha\in I}A_\alpha\subset\bigcap_{\alpha\in I}(X\setminus A_\alpha) \qquad (8)$$

and

$$\bigcap_{\alpha\in I}(X\setminus A_\alpha)\subset X\setminus\bigcup_{\alpha\in I}A_\alpha \qquad (9)\,.$$

To show (8), suppose that $x\in X\setminus\bigcup_{\alpha\in I}A_\alpha$. Then, $x\in X$ and $x\notin\bigcup_{\alpha\in I}A_\alpha$ hence, $x\notin A_\alpha$ for all $\alpha\in I$. Then, $x\in X\setminus A_\alpha$ for all $\alpha\in I$ thus, $x\in\bigcap_{\alpha\in I}(X\setminus A_\alpha)$.
To show (9), suppose that $x\in\bigcap_{\alpha\in I}(X\setminus A_\alpha)$. Then $x\in X\setminus A_\alpha$ for all $\alpha\in I$ that is: $x\in X$ and $x\notin A_\alpha$ for all $\alpha\in I$ and from here $x\notin\bigcup_{\alpha\in I}A_\alpha$. Hence, $x\in X\setminus\bigcup_{\alpha\in I}A_\alpha$.

Cartesian product. Definition 1.6 Let A and B be sets. The *Cartesian product* of A and B, denoted by $A\times B$, is the set of all ordered pairs (a,b), where $a\in A$ and $b\in B$. Thus

$$A\times B=\{(a,b)\,:\,a\in A \text{ and } b\in B\}.$$

For example $\mathbf{R}\times\mathbf{R}$ is the Cartesian plane. For real numbers a and b, the notation (a,b) has two different meanings: It may mean the ordered pair (a,b) or it may mean the open interval $\{x\in\mathbf{R}\,:\,a<x<b\}$. From the context will be clear which meaning is appropriate.

Proposition 1.9 Let A, B, and C be sets. Then

$$A \times (B \cap C) = (A \times B) \cap (A \times C).$$

Proof. First, we show that

$$A \times (B \cap C) \subset (A \times B) \cap (A \times C).$$

Suppose $(x, y) \in A \times (B \cap C)$. Then, $x \in A$ and $y \in B \cap C$,i.e., $y \in B$ and $y \in C$. Then, $(x, y) \in A \times B$ and $(x, y) \in A \times C$ therefore, $(x, y) \in (A \times B) \cap (A \times C)$.
Now, we show that

$$(A \times B) \cap (A \times C) \subset A \times (B \cap C) .$$

Suppose $(x, y) \in A \times B) \cap (A \times C)$. Then, $(x, y) \in A \times B$ and $(x, y) \in A \times C$ hence, $x \in A$, $y \in B$, and $y \in C$. From here, $y \in B \cap C$, therefore, $(x, y) \in A \times (B \cap C)$.

1.3 Functions. The concept of function is basic for mathematics. In this teaching material we consider basic results about functions.

Basic definitions.
Definition 1.7 Let X and Y be two sets. A function (a map) from X into Y is a rule f that assigns to each element x in X **a unique** element $f(x)$ from Y. The set X is called the domain of the function. The set $\{f(x) : x \in X\}$ is called the range of the function.

Definition 1.7 is somewhat not so precise because the term rule is never defined. For this reason an alternative definition is desirable. This next definition identifies a function and its graph.
Definition 1.8 A function from a set X to a set Y is a subset denoted by f of the cartesian product $X \times Y$ such that if $(x, y_1) \in f$ and $(x, y_2) \in f$, then $y_1 = y_2$.
As a notation, we write $f : X \to Y$ to denote that f is a function from X into Y and we often denote the ordered pairs $(x, y) \in f$ by $y = f(x)$.

Two functions f and g are equal, denoted by $f = g$, provided that they have the same domain X and for each element $x \in X$ we have $f(x) = g(x)$.

Definition 1.9 Let f be a function from X to Y. Let $S \subset X$. The direct image of S , denoted by $f(S)$ is defined as

$$f(S) = \{f(s) : s \in S\} .$$

Let $T \subset Y$. The *inverse image* of T, denoted by $f^{-1}(T)$ is defined as

$$f^{-1}(T) = \{x \in X : f(x) \in T\} .$$

For the next examples you may graph the functions to help verify the statements.
Example 1.5 Let $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = x^2$. Hence, $X = \mathbf{R}$ and $Y = \mathbf{R}$.
1. Let S be the set of integers. Then $f(S) = \{0, 1, 4, 9, 16, \ldots\}$.
2. Let $T = \{64\}$. Then $f^{-1}(T) = \{-8, 8\}$.
3. Let $T = \{y : -3 < y < 4\}$. Then $f^{-1}(T) = \{x : -2 < x < 2\}$. Observe that for an element in T less than 0, there are no real numbers that are mapped to that element because $x^2 \geq 0$ for any x. Thus, $f^{-1}((-3, 0)) = \emptyset$.

Example 1.6 Let $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = \sin(x)$. Hence, $X = \mathbf{R}$ and $Y = \mathbf{R}$.
1. Let $S = \{x : 0 \leq x \leq \pi/6\}$. Then $f(S) = \{y : 0 \leq y \leq 1/2\} = [0, 1/2]$.
2. Let $T = \{1\}$. Then, $f^{-1}(T) = \{\pi/2 + 2k\pi : k \text{ is an integer } \}$.
3. Let $T = \{y : 0 \leq y \leq 1\}$. Then, $f^{-1}(T) = \bigcup_{n=-\infty}^{\infty} [2n\pi, (2n+1)\pi]$.
4. Let $T = \{y : \pi/2 \leq y \leq \pi\}$. Then, $f^{-1}(T) = \emptyset$.

Proposition 1.10 Let $f : X \to Y$. Let S and T be subsets of Y. Then

$$f^{-1}(S \cup T) = f^{-1}(S) \cup f^{-1}(T).$$

Proof. **(a)** First, we show that $f^{-1}(S \cup T) \subset f^{-1}(S) \cup f^{-1}(T)$. Suppose, $x \in f^{-1}(S \cup T)$. Then, $f(x) \in S \cup T$ from here, $f(x) \in S$ or $f(x) \in T$. Then, $x \in f^{-1}(S)$ or $x \in f^{-1}(T)$ hence, $x \in f^{-1}(S) \cup f^{-1}(T)$. Thus, $f^{-1}(S \cup T) \subset f^{-1}(S) \cup f^{-1}(T)$.
(b) Next, we show that $f^{-1}(S) \cup f^{-1}(T) \subset f^{-1}(S \cup T)$. Suppose $x \in f^{-1}(S) \cup f^{-1}(T)$. Then $x \in f^{-1}(S)$ or $x \in f^{-1}(T)$ that is $f(x) \in S$ or $f(x) \in T$. From here, $f(x) \in S \cup T$ hence, $x \in f^{-1}(S \cup T)$. Therefore, $f^{-1}(S) \cup f^{-1}(T) \subset f^{-1}(S \cup T)$.
In view of (a) and (b) we conclude that $f^{-1}(S \cup T) = f^{-1}(S) \cup f^{-1}(T)$.

Definition 1.10 A function f from X to Y is *a one-to-one function* (or we write $1-1$ function) if for any pair of distinct points x_1 and x_2 in X, $f(x_1) \neq f(x_2)$ in Y. Equivalently, using the contrapositive: f is one-to-one if from $f(x_1) = f(x_2)$, where $x_1 \in X$ and $x_2 \in X$ follows that $x_1 = x_2$.

Another way to classify a function defined on two sets X and Y vis to look at how much of Y is taken up by the image of X.

Definition 1.11 A function f from X to Y is *onto* Y if $f(X) = Y$. Equivalently, A function f from X to Y is *onto* if for each $y \in Y$ there is $x \in X$ such that $f(x) = y$. In other words, f is onto if its range coincides with Y.

A function f from X to Y is a bijection from X to Y if it is both one-to-one and onto Y.

Example 1.7 The function $f(x) = x^2$ from $\mathbf{R}$ to $\mathbf{R}$ is not one-to-one because $f(-8) = 64 = f(8)$. It is also not onto because for any negative $y \in Y = \mathbf{R}$ there is no $x \in X = \mathbf{R}$ such that $f(x) = y$. We can say also that f is not onto because no negative number is in the range of f.

Example 1.8 Define $h : \mathbf{R} \to \mathbf{R}$ by

$$g(x) = \begin{cases} x - 1 & \text{if} \quad x \geq 0 \\ x + 1 & \text{if} \quad x < 0\,. \end{cases}$$

The function is onto $\mathbf{R}$ but fails to be one-to-one since $g(-1) = g(1) = 0$ and also $g(-1/2) = g(3/2) = 1/2$.

Example 1.9 Define $h : \mathbf{R} \to \mathbf{R}$ by $h(x) = 2x + 7$. Then
1. h is one-to-one function. Suppose $h(x_1) = h(x_2)$. Then $2x_1 + 7 = 2x_2 + 7$ and this obviously implies $x_1 = x_2$.
2. h is onto. Suppose $y \in \mathbf{R}$. Then with $x = (y-7)/2$ we have $h(x) = h((y-7)/2) = 2[(y-7)/2] + 7 = y$.

Example 1.10 Define $f : \mathbf{R} \to \mathbf{R}$ by $f(x) = e^x$. Then show that f is one-to-one but not onto. You may graph the function to see better what you have to do.

The above examples show that all combinations of one-to-one and onto are possible.These two ideas of characterizing a function are independent of one another. Some corollaries of these two properties of a function appear in the next proposition and in the exercises.

Proposition 1.11 Let f be a function from X to Y.
1. For any subsets A and B of X, $f(A \cap B) \subset f(A) \cap f(B)$.
2. If f is one-to-one, then $f(A \cap B) = f(A) \cap f(B)$.
3. For any subsets A and B of X, $f(A \cup B) = f(A) \cup f(B)$.

Proof. 1. Recall that $f(A \cap B) = \{f(x) : x \in A \cap B\}$. Suppose $y \in f(A \cap B)$. Then, there is some $x \in A \cap B$ such that $y = f(x)$. Since, $x \in A \cap B$ we have $x \in A$ and $x \in B$. Hence, $y \in f(A)$ and $y \in f(B)$ and in view of this, $y = f(x) \in f(A) \cap f(B)$. Therefore, $f(A \cap B) \subset f(A) \cap f(B)$.
2. Taking into account 1. we have to show that $f(A) \cap f(B) \subset f(A) \cap B)$. Suppose $y \in f(A) \cap f(B)$. Then $y \in f(A)$ so, there is $a \in A$ such that $y = f(a)$. Analogously, $y \in f(B)$ so, there is $b \in B$ such that $y = f(b)$. Then we have $f(a) = f(b)$ and taking into account that f is one-to-one we conclude that $a = b$ hence, $a = b \in A \cap B$ and in view of this, $y \in f(A \cap B)$.
3. First we show $f(A \cup B) \subset f(A) \cup f(B)$. Suppose $y \in f(A \cup B)$. Then $y = f(x)$ some $x \in A$ or some $xinB$. Then, $y = f(x) \in f(A)$ or $y = f(x) \in f(B)$. Therefore, $y = f(x) \in f(A) \cup f(B)$.
Next, we show that $f(A) \cup f(B) \subset f(A) \cup B)$. Suppose $y \in f(A) \cup f(B)$. Then, $y \in f(A)$ or $y \in f(B)$ that is: There is $a \in A$ such that $y = f(a)$ hence, $y \in f(A) \cup f(B)$ or there is $b \in B$ such that $b \in f(B)$ hence, $y = f(A) \cup f(B)$. Therefore $f(A) \cup f(B) \subset f(A) \cup B)$.

Counterexample that 2. does not hold if f is not one-to-one. Take $f(x) = x^2$, that is not one-to-one on the interval $[-1, 1]$. Take $X = [-1, 1]$ and $Y = [0, 1]$. $A = [-1, 0]$ and $B = [0, 1]$. Then $f(A) = [0, 1]$ $f(B) = [0, 1]$, $A \cap B = \{0\}$. Hence, $f(A \cap B) = f(0) = \{0\} \neq f(A) \cap f(B) = [0, 1]$.

Remark. In Example 1.5 we defined a function $f : \mathbf{R} \to \mathbf{R}$ by $f(x) = x^2$. If we restrict the domain X of f only to the non-negative real numbers, i.e., $f : [0, \infty) \to \mathbf{R}$, then $f(x) = x^2$ is one-to-one on $[0, \infty 0$. If we restrict the range Y of $f(x) = x^2$ to $[0, \infty)$ then $f : [0, \infty) \to [0, \infty)$ is also onto, i.e., $f : [0, \infty) \to [0, \infty)$ is a bijection. Hence, the properties one-to-one and onto for a given function depends essentially on the domain and the range of the function.

Definition 1.12 Let f be a function from X to Y. Let $A \subset X$. The *restriction* of f to A, denoted by $f_{|A}(x)$ is defined by $f_{|A}(x) = f(x)$ for all x in A.
In a similar vein, let g be a function from A to Y, where from $A \subset X$. A function f from X to Y that satisfies $f(x) = g(x)$ for all x in A, i.e., $f_{|A} = g$ is called an extension of g to X (an extension of g from A to X).

Operations with functions. We assume that students are familiar with sum, difference, product, and quotient of two functions. An important operation on function is that of *composition*.

Definition 1.13 (Composition of functions) Let f be a function from X to Y. Let g be a function from Y to Z. The composition of f and g, denoted by $g \circ f$, is a function from X to Z defined by

$$(g \circ f)(x) = g(f(x))$$

for all x in X. The definition can be extended for any finite number of functions.

Proposition 1.12 Let f be a function from X to Y and let g be a function from Y to Z.
1. If both f and g are one-to-one, then $f \circ g$ is one-to-one.
2. If both f and g are onto functions, then $g \circ f$ is onto.
3. If both f and g are bijections, then $g \circ f$ is a bijection.

Proof.
1. Let f and g be one-to-one and let $x_1 \in X$ and $x_2 \in X$, and $x_1 \neq x_2$. Then, $y_1 = f(x_1) \neq y2 = f(x_2)$ because f is one-to-one. Next, $g(f(x_1)) = g(y_1) \neq g(f(x_2)) = g(y_2)$ because $y_1 \neq y_2$ and g is one-to-one. Therefore, $g \circ f$ is one-to-one.

2. Let $z \in Z$. Because g is onto from Y to Z there is $y \in Y$ such that $z = g(y)$. Next, f is onto from X to Y hence, there is some $x \in X$ such that $y = f(x)$. Hence, for each $z \in Z$, there is some $x \in X$ such that $z = g(f(x))$ that is $z = (g \circ f)(x)$. Therefore $g \circ f$ is onto.

3. Follows by 1. and 2.

Example 1.11 Construct a bijection from $\mathbf{R}$ onto the open interval $(0, 1)$.
Solution. Define

$$f : \mathbf{R} \to (0, \infty) \text{ by } f(x) = e^x\,; \qquad g : (0, \infty) \to (1, \infty) \text{ by } g(x) = x + 1\,;$$

$$h : (1, \infty) \to (0, 1) \text{ by } h(x) = \frac{1}{x}\,.$$

Each of the above functions is one-to-one and onto hence, a bijection. Then, by Proposition 1.12

$$(h \circ g \circ f((x) = h(g(f(x))) = f(g(e^x)) = f(e^x + 1) = \frac{1}{1 + e^x}$$

is a bijection (one-to-one and onto) from $\mathbf{R}$ to $(0, 1)$.

Example 1.11′ **(a)** Show that $f(x) = e^x/(1 + e^x$ is a bijection from $\mathbf{R}$ to $(0, 1)$.
(b) Show that $f(x) = 1/(1 + x)$ is a bijection from $(0, \infty)$ to $(0, 1)$.
(c) Construct a bijection from $(0, \infty)$ to the open interval (a, b), where $a < b$.

We conclude this section by showing that a bijection function has a natural function associated with it and that this function in some sense reverses what the original function does.

Proposition 1.13 Let f be a function from X into Y. Then f is a bijection from X onto Y *if and only if* there is a function g from Y onto X such that $(g \circ f)(x) = x$ for all x in X and $(f \circ g)(y) = y$ for all y in Y.

Proof. First, assume that f is a bijection from X onto Y. We define g from Y onto X by $g(y) = x$ if $y = f(x)$, where $x \in X$ and $y \in Y$. Note that f is one-to-one hence, *there is only one* $x \in X$ such that $y = f(x)$ that is $g(y) = x$. Suppose, there are $x_1 \neq x_2$ such that $g(y) = x_1$ and $g(y) = x_2$. Then $f(x_1) = y$ and $f(x_2) = y$ but f is one-to-one and if $x_1 \neq x_2$, then $y = f(x_1) \neq f(x_2) = y$ and get a contradiction. Hence, g is well defined.
Then, for each $x \in X$ we have with $y = f(x)$: $(g \circ f)(x) = g(f(x)) = g(y) = x$. The function f is onto hence, for each $y \in Y$ there is some $x \in X$ such that $y = f(x)$ and $(f \circ g)(y) = f(g(y)) = f(x) = y$.
We show that g is also a bijection. For each $x \in X$ there is $y \in Y$ such that $g(y) = x$ that is $y = f(x)$ hence, g is onto. If $y_1 \neq y_2$ with $g(y_1) = x_1$ and $g(y_2) = x_2$ (that is $f(x_1) = y_1$ and $f(x_2) = y_2$), then $x_1 \neq x_2$. Suppose $x_1 = x_2$, then $y_1 = f(x_1) = f(x_2) = y_2$ and we get a contradiction. Hence g is one-to-one.
Next, assume that such a function g exists. We must show that f is a bijection. Suppose $y \in Y$. Then, $(f \circ g)(y) = f(g(y)) = y$. Denote $x = g(y) \in X$. Then, $(f \circ g)(y) = f(g(y)) = f(x) = y$ hence, for each $y \in Y$ we find $x \in X$ such that $y = f(x)$. From here, f is onto.
Suppose $x_1 \in X$ and $x_2 \in X$ such that $x_1 \neq x_2$ and $f(x_1 = f(x_2)$. Then, $x_1 = g(f(x_1)) = g(f(x_2)) = x_2$ and we get a contradiction hence, $f(x_1) \neq f(x_2)$ therefore, f is one-to-one. Summing up, f is a bijection.

Remark. The function g that has been constructed in Proposition 1.13 is called the *inverse* of f and is denoted by $f^{-1} = g$. Restating the Proposition 1.13 in terms of the notation f^{-1} : Let f be a function from X into Y. Then f is a bijection from X onto Y *if and only if* there is a function f^{-1} from Y onto X such that $(f^{-1} \circ f)(x) = x$ for all x in X and $(f \circ f^{-1})(y) = y$ for all y in Y.

Mathematical induction. In this course mathematical induction will be used often to prove statements. Recall that are a given statement is either true or false but noth both, i.e., it can not be true and false.
Mathematical induction. Let $p(n)$ be a statement for each $n \in \mathbf{N}$. Assume that
1. $p(1)$ is true.
2. For each $k \geq 1$, k positive integer, if $p(k)$ is true, then $p(k+1)$ is also true.
Then $p(n)$ is truer for all $n \in \mathbf{N}$.

In the assumption 2, "$p(k)$ is true" is called the *induction hypothesis*. Intuitively, $p(1)$ true implies $p(2)$ is true by assumption 2.; $p(2)$ is true implies that $p(3)$ is true by assumption 2; and so on.

Before illustrating the method of mathematical induction, we examine the relationship between mathematical induction and the following concept, which will be assumed as an axiom in our next considerations in this course.

Axiom. $\mathbf{N}$ is *Well-Ordered*. Every nonempty subset of $\mathbf{N}$ has a least element. That is: If $A \subset \mathbf{N}$ and $A \neq \emptyset$, then there is an $a_0 \in A$ such that $a_0 \leq a$ for all $a \in A$.

Theorem 1.3 $\mathbf{N}$ is well-ordered if and only if mathematical induction is true (holds).
Proof. First, assume that $\mathbf{N}$ is well-ordered. For each positive integer n, let $p(n)$ be a statement satisfying the assumptions 1. and 2. of the mathematical induction. We want to show that $p(n)$ is true for all positive integer n. *Suppose to the contrary, that this is false.* Let $A \subset \mathbf{N}$ be the set of positive integers for which $p(n)$, $n \in A$ is false. Since $\mathbf{N}$ is well-ordered, A has a least element a_0. By assumption $p(1)$ is true so, $a_0 > 1$. Therefore $a_0 - 1$ is positive integer and $p(a_0 - 1)$ is true since a_0 is the least positive integer for which the statement $p(n)$ is false. By assumption 2. $p((a_0 - 1) + 1) = p(a_0)$ is also true, which is a contradiction.Hence, $p(n)$ is true (holds) for every positive integer n.

Next, suppose that the mathematical induction holds. We wish to show that $\mathbf{N}$ is well-ordered. Let A be a non-empty set subset of $\mathbf{N}$ and we have to show that A has a least element. *Suppose to the contrary that A has no least element.* For each $n \in \mathbf{N}$, let $p(n)$ be the statement

$$A \cap \{1, 2, \ldots, n\} = \emptyset\,.$$

Suppose $p(1)$ is false. Then, $A \cap \{1\} \neq \emptyset$ and A has a least element-namely, 1 and this is a contradiction to [A does not have a least element]. Hence, $p(1)$ is true. Let $k \geq 1$, positive integer and suppose that $p(k)$ is true, thus

$$A \cap \{1, 2, \ldots, k\} = \emptyset\,.$$

Suppose that $p(k+1)$ is false. Then

$$A \cap \{1, 2, \ldots, k, k+1\} \neq \emptyset$$

and from here, A has a least element - $(k+1)$ which is again a contradiction to [A does not have a least element]. Therefore $p(k+1)$ is true.
Then, by the mathematical induction $p(n)$ is true for all positive integer n, i.e., for all $n \in \mathbf{N}$. In view of this, $A = \emptyset$ and this is a contradiction to [A is a non-empty set]. Therefore , A has a least element hence, $\mathbf{N}$ is well-ordered.

Example 1.13 For each $n \in \mathbf{N}$

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}\,.$$

Solution. For each $n \in \mathbf{N}$, let $p(n)$ be the statement

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}\,.$$

Then, $p(1)$ is true since obviously,

$$1^2 = \frac{(1)(1+1)(2+1)}{6}\,.$$

Let $k \geq 1$ and assume that $p(k)$ is true, i.e.,

$$1^2 + 2^2 + 3^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}\,.$$

We need to show that $p(k+1)$ is true, i.e.,

$$1^2 + 2^2 + 3^2 + \cdots + k^2 + (k+1)^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$$

that is

$$1^2 + 2^2 + 3^2 + \cdots + k^2 + (k+1)^2 = \frac{(k+1)((k+2)(2k+3)}{6}\,.$$

By the induction hypothesis we have

$$1^2 + 2^2 + 3^2 + \cdots + k^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

$$= (k+1)\left[\frac{2k^2+k}{6} + (k+1)\right] = (k+1)\left[\frac{2k^2+7k+6}{6}\right]$$

$$= (k+1)\left[\frac{(2k^2+3k)+(4k+6)}{6}\right] = (k+1)\left[\frac{k(2k+3)+2(2k+3)}{6}\right]$$

$$= (k+1)\left[\frac{(2k+3)(k+2)}{6}\right] = \frac{(k+1)((k+2)(2k+3)}{6}\,.$$

Example 1.14 For each $n \in \mathbf{N}$, $2^n \geq n+1$.
Solution. Let $p(n)$ be the statement $2^n \geq n+1$ for each positive integer n. Obviously, $2^1 \geq 1+1$ hence, $p(1)$ is true. Next suppose that $p(k)$ holds, i.e., $2^k \geq k+1$ and we have to show that $p(k+1)$ holds, i.e., $2^{k+1} \geq k+2$. Observe that

$$2^{k+1} = 2^k \cdot 2 \geq (k+1)\cdot 2 = 2k+2 \geq k+2$$

since $2k \geq k$. Thus, if $p(k)$ is true, then $p(k+1)$ is also true. Hence by the mathematical induction, $p(n)$ holds for all n in $\mathbf{N}$.
Example 1.14 For each $n \in \mathbf{N}$, 9 divides $n^3+(n+1)^3+(n+2)^3$, where divides means with 0 remainder.
Solution. Let $p(n)$ be the statement 9 divides $n^3+(n+1)^3+(n+2)^3$, for each positive integer n. Obviously, 9 divides $1^3+(1+1)^3+(1+2)^3 = 9+27 = 36$ hence, $p(1)$ is true. We assume $p(k)$ is true for some $k \geq 1$, i.e., 9 divides $k^3+(k+1)^3+(k+2)^3$. We must show that $p(k+1)$ is true, i.e., 9 divides $(k+1)^3+(k+2)^3+(k+3)^3$. Observe, that

$$(k+1)^3+(k+2)^3+(k+3)^3 = [k^3+(k+1)^3+(k+2)^3]+[(k+3)^3-k^3]$$

$$= [k^3+(k+1)^3+(k+2)^3]+[(k+3)-k][(k+3)^2+k(k+3)+k^2]$$

$$= [k^3+(k+1)^3+(k+2)^3]+3(3k^2+9k+9) = [k^3+(k+1)^3+(k+2)^3]+9(k^2+3k+3)\,.$$

By the induction hypothesis 9 divides $k^3+(k+1)^3+(k+2)^3$ and obviously, $9(k^2+3k+3)$ is a multiple of 9 hence, divisible by 9. In view of this, $p(k+1)$ is also true. By the principle of mathematical induction $p(n)$ is true for all positive integer n that is: 9 divides $n^3+(n+1)^3+(n+2)^3$ for each positive integer n.

Remark. Sometimes mathematical induction is stated as follows: Let A be a subset of $\mathbf{N}$ satisfying
1. $1 \in A$.
2. If $k \in A$, then $k+1 \in A$.
Then $A = \mathbf{N}$.
We see that this is equivalent to the previous version of mathematical induction by letting $A = \{n : p(n) \text{ is true}\}$.

Remark. the following represents misapplication (misuse) of the mathematical induction. Find the error.
For each n in $\mathbf{N}$ *let $p(n)$ be the statement: Any set of n horses are all of the same color. $p(1)$ is obviously true, we have only one horse. Let $k \geq 1$ and assume that $p(k)$ is true. That is: Assume that any set of k horses are all of the same color. We want to show that $p(k+1)$ is true that is: Any set of $(k+1)$ horses is of the same color. Let $X = [x_1, x_2, \ldots, x_{k+1}]$ be a set of $k+1$ horses. Since $[x_1, x_2, \ldots, x_k]$ is a set of k horses, by the induction hypothesis, these are all of the same color. Since $[x_2, x_3, \ldots, x_{k+1}]$ is a set of k horses, by the induction hypothesis, these are all of the same color. Thus all $(k+1)$ horses are of the same color. Therefore $p(n)$ is true for all positive integer n.*
Hint. For which k does the argument fails?