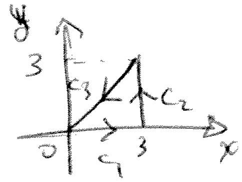
MAT 2322 - 3X - Assignment # 3

Solutions:

1. a/. C_1 is given by $\vec{r}_1 = t\,\vec{i}$, $0 \le t \le 3$

C_2 is given by $\vec{r}_2 = 3\vec{i} + t\vec{j}$, $0 \le t \le 3$

C_3 is given by $\vec{r}_3 = (3-t)\,\vec{i} + (3-t)\,\vec{j}$, $0 \le t \le 3$.



$$\oint_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r} + \int_{C_3} \vec{F}\cdot d\vec{r}$$

$$= \int_0^3 (-t^2\vec{i} + t^2\vec{j})\cdot\vec{i}\,dt + \int_0^3 \left((t^2-9)\,\vec{i} + (9+t^2)\,\vec{j}\right)\cdot\vec{j}\,dt$$

$$+ \int_0^3 \left(0\,\vec{i} + 2(3-t)^2\,\vec{j}\right)\cdot(-\vec{i}-\vec{j})\,dt$$

$$= \boxed{9}$$

b/ By Green's Thm: $\oint_C \vec{F}\cdot d\vec{r} = \iint_D (Q_x - P_y)\,dA$,

where $D = \{(x,y) \mid 0 \le x \le 3,\ 0 \le y \le x\} \overset{\text{or}}{=} \{(x,y) \mid 0 \le y \le 3,\ y \le x \le 3\}$.

$$\Leftrightarrow \oint_C \vec{F}\cdot d\vec{r} = \int_0^3 \int_0^x \left[(9-6y) - (y^2 - 2y^2)\right] dy\,dx = \int_0^3 \int_0^x (y^2 - 6y + 9)\,dy\,dx$$

$$= \boxed{9}$$

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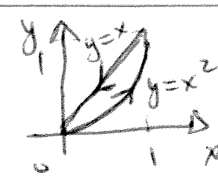
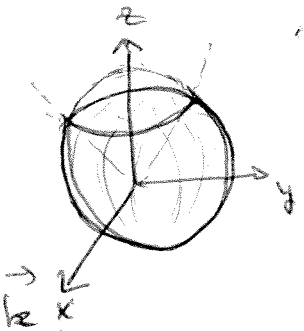
2. By Green's Thm: $\oint_C \vec{F}\cdot d\vec{r} = \iint_D (Q_x - P_y)\,dA$



$$= \int_0^1 \int_y^{\sqrt{y}} (-x)\,dx\,dy = \boxed{-\frac{1}{12}}.$$ Here

$D = \{(x,y) \mid 0 \le y \le 1,\ y \le x \le \sqrt{y}\} \overset{\text{or}}{=} \{(x,y) \mid 0 \le x \le 1,\ x^2 \le y \le x\}$.



3. • Parametrization:

$x = \sqrt{2}\sin(\phi)\cos(\theta)$, $y = \sqrt{2}\sin(\phi)\sin(\theta)$, $z = \sqrt{2}\cos(\phi)$

$\frac{\pi}{4} \leq \phi \leq \pi$, $0 \leq \theta \leq 2\pi$.

$\Rightarrow \vec{r}(\phi, \theta) = \sqrt{2}\sin(\phi)\cos(\theta)\vec{i} + \sqrt{2}\sin(\phi)\sin(\theta)\vec{j} + \sqrt{2}\cos(\phi)\vec{k}$

& $\vec{r}_\phi = \sqrt{2}\cos(\phi)\cos(\theta)\vec{i} + \sqrt{2}\cos(\phi)\sin(\theta)\vec{j} - \sqrt{2}\sin(\phi)\vec{k}$

$\vec{r}_\theta = -\sqrt{2}\sin(\phi)\sin(\theta)\vec{i} + \sqrt{2}\sin(\phi)\cos(\theta)\vec{j} + 0\vec{k}$

So $\vec{r}_\phi \times \vec{r}_\theta = 2\sin^2(\phi)\cos(\theta)\vec{i} + 2\sin^2(\phi)\sin(\theta)\vec{j} + 2\cos(\phi)\sin(\phi)\vec{k}$.

& $|\vec{r}_\phi \times \vec{r}_\theta| = 2\sin(\phi)$.

• The surface area is then $A = \int_0^{2\pi}\int_{\pi/4}^{\pi} |\vec{r}_\phi \times \vec{r}_\theta| \, d\phi \, d\theta$

$= \int_0^{2\pi}\int_{\pi/4}^{\pi} 2\sin(\phi)\, d\phi \, d\theta$

$= \boxed{4\pi\left(1 + \frac{\sqrt{2}}{2}\right)}$

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4. • Parametrization: $\vec{r}(\phi, \theta) = \sin(\phi)\cos(\theta)\vec{i} + \sin(\phi)\sin(\theta)\vec{j} + \cos(\phi)\vec{k}$;

$0 \leq \phi \leq \pi$, $0 \leq \theta \leq 2\pi$.

• As in Question 3, $|\vec{r}_\phi \times \vec{r}_\theta| = \sin(\phi)$

• Surface integral: $\iint_S f \cdot dS = \int_0^{2\pi}\int_0^{\pi} (\sin^2(\phi)\cos^2(\theta)) \cdot (\sin(\phi))\, d\phi\, d\theta$

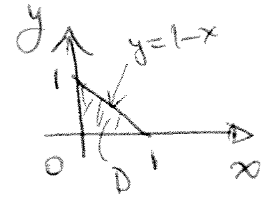
$= \left(\int_0^{2\pi} \cos^2(\theta)\, d\theta\right)\left(\int_0^{\pi} \sin^3(\phi)\, d\phi\right)$

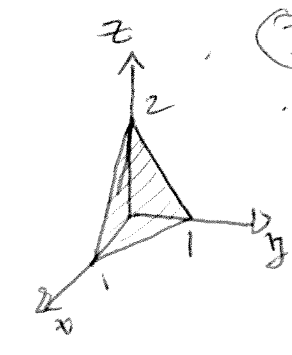
$= \boxed{\frac{4\pi}{3}}$

5. • S is the graph of $z = 2 - 2x - 2y$.

The projection of S in the xy-plane is

$D = \{(x,y) \mid 0 \le x \le 1,\ 1 \le y \le 1-x\}$





• A parametrization of S is

$\vec{r}(x,y) = x\,\vec{i} + y\,\vec{j} + (2-2x-2y)\,\vec{k}$, $0 \le x \le 1$, $0 \le y \le 1-x$.

$\Rightarrow |\vec{r}_x \times \vec{r}_y| = (1 + z_x^2 + z_y^2)^{1/2} = \sqrt{9} = 3$

see lecture notes

• Surface integral: $\iint_S f\,dS = \int_0^1 \int_0^{1-x} f(x,y,2-2x-2y) \cdot (3)\, dy\, dx$

$= \int_0^1 \int_0^{1-x} (2-x-y)(3)\, dy\, dx = \boxed{2}$

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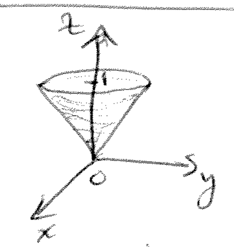
7. • Parametrization: $\vec{r}(\theta, z) = z\cos(\theta)\,\vec{i} + z\sin(\theta)\,\vec{j} + z\,\vec{k}$

$0 \le z \le 1$, $0 \le \theta \le 2\pi$



since the cone is $z = \sqrt{x^2+y^2} = r$.

(We could use $\vec{r}(r,\theta) = r\cos(\theta)\,\vec{i} + r\sin(\theta)\,\vec{j} + r\,\vec{k}$, $0 \le r \le 1$, $0 \le \theta \le 2\pi$)

$\Rightarrow \vec{r}_\theta \times \vec{r}_z = (-z\sin(\theta)\,\vec{i} + z\cos(\theta)\,\vec{j}) \times (\cos(\theta)\,\vec{i} + \sin(\theta)\,\vec{j} + \vec{k})$

$\vec{r}_\theta \times \vec{r}_z = z\cos(\theta)\,\vec{i} + z\sin(\theta)\,\vec{j} - z\,\vec{k}$ oriented downward/outward.

Also $\vec{F}(\vec{r}(\theta,z)) = z^2\cos(\theta)\sin(\theta)\,\vec{i} - z\,\vec{k}$

• Surface integral: $\iint_S \vec{F} \cdot d\vec{S} = \int_0^{2\pi} \int_0^1 (z^2\cos(\theta)\sin(\theta)\,\vec{i} - z\,\vec{k}) \cdot (z\cos(\theta)\,\vec{i} + z\sin(\theta)\,\vec{j} - z\,\vec{k})\, dz\, d\theta$

$= \int_0^{2\pi} \int_0^1 z^2(\cos^2(\theta)\sin(\theta) + 1)\, dz\, d\theta$

$= \boxed{\dfrac{2\pi}{3}}$