MATH2004 Notes - By Eric Hua

Contents

Pre-knowledge

1. Trig Identities:

   - $\cos^2 x = \dfrac{1+\cos 2x}{2}$, $\sin^2 x = \dfrac{1-\cos 2x}{2}$, $2\sin x \cos x = \sin 2x$.
   - $\sin a \sin b = \dfrac{\cos(a-b)-\cos(a+b)}{2}$, $\cos a \cos b = \dfrac{\cos(a+b)+\cos(a-b)}{2}$,
   - $\sin a \cos b = \dfrac{\sin(a+b)+\sin(a-b)}{2}$.

2. Integration:

   - Fundamental Theorem of Calculus: $\displaystyle\int_a^b f(x)dx = F(b) - F(a)$, where $F'(x) = f(x)$.
   - Integration by parts: $\displaystyle\int u(x)v'(x)dx = u(x)v(x) - \int u'(x)v(x)dx$, or $\displaystyle\int udv = uv - \int vdu$.
   - Integration by substitution: $\displaystyle\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$, where $u = g(x)$.
   - Trig integral: $\displaystyle\int \sin^m x \cos^n x dx$:
     - If m is odd, then let $u = \cos x$.
     - If n is odd, then let $u = \sin x$.
     - If m and n are even, then use half-angle formula.
   - Trigonometric substitutions:
     - $\sqrt{a^2 - x^2}$: Let $x = a\sin\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$;
     - $\sqrt{a^2 + x^2}$: Let $x = a\tan\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$;
     - $\sqrt{x^2 - a^2}$: Let $x = a\sec\theta$, $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$.

Chapter 1: Vectors

1.1–1.6. Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$

Vector has two components: Magnitude and direction.

Algebraic representation of vectors in the plane $\mathbb{R}^2$:

- Vectors in $\mathbb{R}^2$: $\vec{v} = \mathbf{v} = (a, b) = \begin{bmatrix} a \\ b \end{bmatrix}$, zero vector $\vec{0} = (0, 0)$.
- Length (norm, magnitude) $|(a, b)| = \sqrt{a^2 + b^2}$.
- Sum: Let $\vec{u} = (u_1, u_2)$, $\vec{v} = (v_1, v_2)$, then $\vec{u} + \vec{v} = (u_1 + v_1, u_2 + v_2)$.
- Scalar multiple: Let $\vec{u} = (u_1, u_2)$, c be a scalar, then $c\vec{u} = (cu_1, cu_2)$.
- Distance: Let $\vec{u} = (u_1, u_2)$, $\vec{v} = (v_1, v_2)$, then $d(\vec{u}, \vec{v}) = |\vec{u} - \vec{v}|$.
- unit vector: $|\vec{u}| = 1$.
- Standard basis vectors: $\vec{i} = (1, 0), \vec{j} = (0, 1)$. Position vectors can be expressed in terms of standard basis vectors: $(a, b) = a\vec{i} + b\vec{j}$.

Algebraic representation of vectors in $\mathbb{R}^3$:

- Vectors in $\mathbb{R}^3$: $\vec{v} = (a, b, c) = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, zero vector $\vec{0} = (0, 0, 0)$.
- Length (norm, magnitude) $|(a, b, c)| = \sqrt{a^2 + b^2 + c^2}$.
- Sum: Let $\vec{u} = (u_1, u_2, u_3)$, $\vec{v} = (v_1, v_2, v_3)$, then $\vec{u} + \vec{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$.

- Scalar multiple: Let $\vec{u} = (u_1, u_2, u_3)$, c be a scalar, then $c\vec{u} = (cu_1, cu_2, cu_3)$.

- Distance: Let $\vec{u} = (u_1, u_2, u_3)$, $\vec{v} = (v_1, v_2, v_3)$, then $d(\vec{u}, \vec{v}) = |\vec{u} - \vec{v}|$.

- unit vector: $|\vec{u}| = 1$.

- Standard basis vectors: $\vec{i} = (1, 0, 0), \vec{j} = (0, 1, 0), \vec{k} = (0, 0, 1)$. Position vectors can be expressed in terms of standard basis vectors: $(a, b, c) = a\vec{i} + b\vec{j} + c\vec{k}$.

Properties: Let c, d be scalars.

- $\vec{u} + \vec{v} = \vec{v} + \vec{u}$, $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$, $\vec{u} + \vec{0} = \vec{u}$, $\vec{u} + (-\vec{u}) = \vec{0}$;

- $(cd)\vec{u} = c(d\vec{u})$, $(c + d)\vec{u} = c\vec{u} + d\vec{u}$, $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$;

- $1\vec{u} = \vec{u}$, $(-1)\vec{u} = -\vec{u}$, $0\vec{u} = \vec{0}$,

- $\vec{u} // \vec{v} \Leftrightarrow \vec{v} = c\vec{u}$.

Example 1. *Given a vector $\vec{v}$, find the unit vector $\vec{u}$ which has the same direction as $\vec{v}$:*

(i) $\vec{v} = (-3, 4)$, (ii) $\vec{v} = (1, 2, -2)$.

Solution: (i)

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{(-3, 4)}{5} = (-\frac{3}{5}, \frac{4}{5}).$$

(ii)

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{(1, 2, -2)}{3} = (\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}).$$

1.7-1.10. The Dot Product, Cross Product, and Applications

- Dot product: Let $\vec{u} = (u_1, u_2, u_3)$, $\vec{v} = (v_1, v_2, v_3)$, then $\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_3v_3$.

- Angle: Let θ be the angle between $\vec{u}$ and $\vec{v}$ which satisfies $0 \leq \theta \leq \pi$, then $\cos\theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}|\,|\vec{v}|}$.

- Orthogonal: $\vec{u} \perp \vec{v}$ if $\vec{u} \cdot \vec{v} = 0$.

- Direction angles to the three axis and direction cosines of vectors.:

$$\cos\alpha = \frac{\vec{u}\cdot\vec{i}}{|\vec{u}|\,|\vec{i}|}, \quad \cos\beta = \frac{\vec{u}\cdot\vec{j}}{|\vec{u}|\,|\vec{j}|}, \quad \cos\gamma = \frac{\vec{u}\cdot\vec{k}}{|\vec{u}|\,|\vec{k}|}.$$

  They satisfy

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1, \quad \frac{\vec{u}}{|\vec{u}|} = (\cos\alpha, \cos\beta, \cos\gamma).$$

- Projection: The projection of $\vec{u}$ onto $\vec{v}$ is

$$proj_{\vec{v}}\vec{u} = \left(\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\right)\vec{v}, \quad comp_{\vec{v}}\vec{u} = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|}.$$

- Work done by the force $\vec{F}$ and the displacement vector $\vec{d}$ is: $W = \vec{F}\cdot\vec{d}$.

Example 2. *Let $\vec{u} = (1, 2, -2)$, $\vec{v} = (-2, -2, 1)$, Find the cosine of the angle between $\vec{u}$ and $\vec{v}$.*

Solution:

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}|\,|\vec{v}|} = \frac{-8}{9}.$$

Properties: Let c be a scalar.

- $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$
- $\vec{w}\cdot(\vec{u}+\vec{v}) = \vec{w}\cdot\vec{u} + \vec{w}\cdot\vec{v}$
- $c(\vec{u}\cdot\vec{v}) = (c\vec{u})\cdot\vec{v} = \vec{u}\cdot(c\vec{v})$
- $\vec{u}\cdot\vec{0} = 0$
- $\vec{u}\cdot\vec{u} = |\vec{u}|^2$.
- Cross product: Let $\vec{u} = (u_1, u_2, u_3)$, $\vec{v} = (v_1, v_2, v_3)$, then

$$\vec{u}\times\vec{v} = (u_2v_3 - u_3v_2, +u_3v_1 - u_1v_3, u_1v_2 - u_2v_1).$$

- Orthogonal: $\vec{u}\times\vec{v}\perp\vec{u}$, $\vec{u}\times\vec{v}\perp\vec{v}$.

Example 3. *Find a vector that is orthogonal to both $\vec{u} = (1, 2, -1)$, $\vec{v} = (0, 2, 3)$.*

Solution: Any scalar multiple of $\vec{u}\times\vec{v} = (8, -3, 2)$.

Properties: Let c be a scalar.

- $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$
- $\vec{w} \times (\vec{u} + \vec{v}) = \vec{w} \times \vec{u} + \vec{w} \times \vec{v}$
- $(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$
- $c(\vec{u} \times \vec{v}) = (c\vec{u}) \times \vec{v} = \vec{u} \times (c\vec{v})$
- $\vec{u} \times \vec{0} = \vec{0}$
- $\vec{u} \times \vec{u} = \vec{0}$
- $|\vec{u} \times \vec{v}| = |\vec{u}|\,|\vec{v}|\,\sin\theta$, where θ is the angle between $\vec{u}$ and $\vec{v}$
- $|\vec{u} \times \vec{v}|$ is the area of the parallelogram determined by $\vec{u}$ and $\vec{v}$.
- $\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$
- $\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$.

Example 4. *Find the area of the parallelogram determined by* $\vec{u} = (1, 2, -1)$, $\vec{v} = (0, 2, 3)$.

Solution: $A = |\vec{u} \times \vec{v}| = |(8, -3, 2)| = \sqrt{77}$.

Example 5. *Find the area of the triangle with vertices* $P(1, 2, 3)$, $Q(-3, 2, 1)$, *and* $R(2, 4, 5)$.

Solution: $\vec{PQ} = Q - P = (-4, 0, -2)$, $\vec{PR} = R - P = (1, 2, 2)$.

$$A = \frac{1}{2}|\vec{PQ} \times \vec{PR}| = \frac{1}{2}|(4, 6, -8)| = \frac{1}{2}\sqrt{29}.$$

The Scalar Triple Product:

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

- The volume V of the parallelepiped formed by the three vectors $\vec{u}, \vec{v}, \vec{w}$ is given by $V = |\vec{u} \cdot (\vec{v} \times \vec{w})|$.
- the three vectors $\vec{u}, \vec{v}, \vec{w}$ are coplanar iff $\vec{u} \cdot (\vec{v} \times \vec{w}) = 0$.

Example 6. *Find the volume* V *of the parallelepiped formed by the three vectors* $\vec{u} = (1, 2, 1), \vec{v} = (0, 1, -1), \vec{w} = (1, 1, 0)$.

Solution: $\vec{v} \times \vec{w} = (1, -1, -1)$,
$\vec{u} \cdot (\vec{v} \times \vec{w}) = (1, 2, 1) \cdot (1, -1, -1) = -2$.
$V = |-2| = 2$.

Chapter 2: Curves and Surfaces

2.1-2.5. Lines and Planes

Line:

A line is determined by a point P and a vector $\vec{v}$ (direction vector) parallel to the line:

$$\vec{r}(t) = P + t\vec{v}, \quad t \in \mathbb{R},$$

where $\vec{r}(t) = (x, y)$ in 2D, and (x, y, z) in 3D.

Example 7. *For L in 2D: $ax + by = c$. A direction vector is $(-b, a)$.*

Example 8. *Find the equation of the line through $P(1, 2, 3)$ and $Q(3, 1, 1)$.*

Solution: $\vec{v} = Q - P = (2, -1, -2)$.

$$(x, y, z) = (1, 2, 3) + t(2, -1, -2), \quad t \in \mathbb{R}.$$

Relation between two lines L_1 and L_2

- parallel
- intersected
- skewed

Example 9. *Show that the intersection between $L_1 : x = 2 - t, y = -1 - t, z = 4 - t$ and $L_2 : x = 5 - 2s, y = -s, z = 1 + s$ is $(1, -2, 3)$.*

Solution: From "$x = x$" and "$y = y$" we have $2 - t = 5 - 2s, -1 - t = -s \Rightarrow t = 1, s = 2$.

By L_1, $z = 3$; by L_2, $z = 3$. Two z values are equal.

Thus the two lines have an intersection: (1,-2,3).

Example 10. *Show that the two lines $L_1 : x = 1 + t, y = -2 + 3t, z = 4 - t; L_2 : x = 2s, y = 3 + s, z = -3 + 4s$ are skew lines.*

Solution: (1) The two direction vectors are $v_1 = (1, 3, -1)$ and $v_2 = (2, 1, 4)$. They are not parallel.

(2) No intersection: From "$x = x$" and "$y = y$" we have $1 + t = 2s, -2 + 3t = 3 + s \Rightarrow s = 1.6, t = 2.2$. By L_1, $z = 1.8$; by L_2, $z = 3.4$.

Line segment:

A line segment between two points $P(x_0, y_0, z_0)$ and $Q(x_1, y_1, z_1)$:

$$\vec{r}(t) = (1-t)\vec{P} + t\vec{Q}, \ 0 \leq t \leq 1.$$

Example 11. *Find the equation of the line segment between* $P(1, 2, 3)$ *and* $Q(3, 1, 1)$.

Plane:

A plane Π is determined by a point and a normal vector $\vec{n} = (a, b, c)$ which is perpendicular to the plane. Let $P(p_1, p_2, p_3)$ be a point on the plane:

$$ax + by + cz = d, \quad d = ap_1 + bp_2 + cp_3.$$

Example 12. *Find the equation of the plane through three points* $P(1, 2, 3)$, $Q(-3, 2, 1)$, *and* $R(2, 4, 5)$.

Solution: $\vec{PQ} = Q - P = (-4, 0, -2)$, $\vec{PR} = R - P = (1, 2, 2)$.

$$\vec{n} = \vec{PQ} \times \vec{PR} = (4, 6, -8) = 2(2, 3, -4).$$

Thus

$$4x + 6y - 8z = -8, \Rightarrow 2x + 3y - 4z = -4.$$

Example 13. *Find the intersection between the line* $L: x = 1 + t, y = -2 + 3t, z = 4 - t$ *and the plane* $3x + 5y + 8z = 5$.

Solution: Substitute the line into the plane:

$$3(1 + t) + 5(-2 + 3t) + 8(4 - t) = 5, \quad 10t = -20, \quad t = -2.$$

The intersection is: (-1,-8,6).

Example 14. *Find the plane containing* $L_1: x = 1 + t, y = -2 + 3t, z = 4 - t$ *and* $L_2: x = 2s, y = 3 + s, z = -3 + 3s$.

Solution:

$$\vec{n} = (1, 3, -1) \times (2, 1, 3) = (10, -5, -4).$$

Let the plane be $10x - 5y - 4z + d = 0$. Sub $(0, 3, -3)$, $d = 3$. Thus the plane is: $10x - 5y - 4z + 3 = 0$.

Angle between two planes:

Two planes are parallel if their normal vectors are parallel. The angle between two planes is defined as the angle between their normal vectors:

$$\cos\theta = \frac{\vec{n_1} \cdot \vec{n_2}}{|\vec{n_1}|\,|\vec{n_2}|}.$$

Example 15. *Find the angle between two planes* $x - 3y - 2z = 3$ *and* $3x + 5y + 8z = 1$.

Solution: $\cos\theta = \frac{-2}{\sqrt{7}}, \Rightarrow \theta = 180^o - 41^o = 139^o$.

Intersection between two planes:

If the two planes with normal vectors $\vec{n_1}$ and $\vec{n_2}$ are not parallel, then the intersection is a line with direction vector $\vec{n_1} \times \vec{n_2}$.

Example 16. *Find the intersection between two planes* $x-3y-2z = 2$ *and* $2x+y+3z = 1$.

Solution: (1) $\vec{n_1} \times \vec{n_2} = (-7, 7, 7) = 7(-1, 1, 1)$.

(2) To find one intersection point, we let $z = 0$. Then $x = 5/7, y = -3/7$. So the parametric equation of the line is:

$$x = 5/7 - t, y = -3/7 + t, z = t.$$

2.6 Rotations and Translations in the Plane

If we rotate a point $P(x, y)$ counter-clockwise by an angle θ to a new point $P'(x', y')$, then

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix}, \quad A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = A^{-1} \begin{pmatrix} x' \\ y' \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

A passive rotation is a rotation of coordinate axes by an angle θ counter-clockwise, from the $xy-$plane to $x'y'$-plane. Then

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = A^{-1} \begin{pmatrix} x \\ y \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

Transformations with a translated origin: If we move the origin $(0,0)$ of the xy-plane to a new point (h,k) which is the origin of the $x'y'$-plane, and rotate the line segment L joining (h,k) and (x,y) about the point (h,k), then

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = A \begin{pmatrix} x-h \\ y-k \end{pmatrix}, \quad A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Example 17. *Find the coordinates (x', y') of the point $(-3,3)$ under the change of the origin **O=(1,5)** by a counterclockwise rotation about the new origin with the angle $\theta = \frac{\pi}{3}$.*

Solution:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = A \begin{pmatrix} x-h \\ y-k \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x-h \\ y-k \end{pmatrix}$$

$$= \begin{pmatrix} \cos\frac{\pi}{3} & -\sin\frac{\pi}{3} \\ \sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{pmatrix} \begin{pmatrix} -3-1 \\ 3-5 \end{pmatrix} = \begin{pmatrix} -2+\sqrt{3} \\ -2\sqrt{3}-1 \end{pmatrix}$$

2.7-2.8. Parametric Curves

Some curves:

- Parabola: The set of all points in a plane that are equidistant from a fixed point (called the focus) and a fixed line (called directrix). An equation of the parabola with focus $(0,p)$ and directrix $y=-p$ is $(x-x_0)^2 = 4p(y-y_0)$.
- Ellipse: The set of all points in a plane the sum of whose distances from two fixed points (called the foci) is a constant. An equation of the ellipse with foci $(\pm c, 0)$ and vertices $(\pm a, 0)$ is $\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1$, $a \geq b > 0$, $c^2 = a^2 - b^2$.
- Hyperbola: The set of all points in a plane the difference of whose distances from two fixed points (called the foci) is a constant. An equation of the ellipse with foci $(\pm c, 0)$ and vertices $(\pm a, 0)$ is $\frac{(x-x_0)^2}{a^2} - \frac{(y-y_0)^2}{b^2} = 1$, $a \geq b > 0$, $c^2 = a^2 + b^2$.

Instead of defining y in terms of x, $y = f(x)$, we define both x and y in terms of a third variable called a parameter as follows: $x = f(t), y = g(t)$. This third variable t is called a parameter. The collection of points $(x, y) = (f(t), g(t))$ that we get by letting t be all possible values is the graph of the parametric equations and is called the parametric curve.

Plane curve (parametric curve): The set of ordered pairs (points)

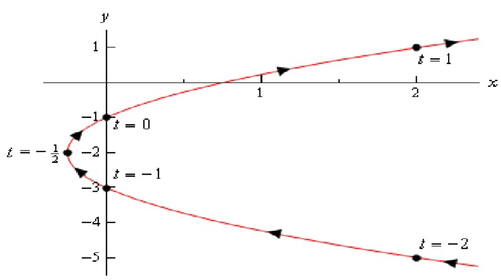
$$(x, y) = (f(t), g(t)), \quad a \leq t \leq b,$$

where f and g are continuous functions, $(f(a), g(a))$ is called initial point, $(f(b), g(b))$ is called terminal point.

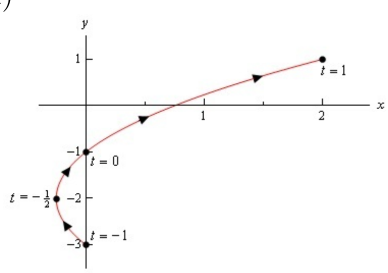
Example 18. *Sketch the parametric curve for the following set of parametric equations.* $x = t^2 + t, y = 2t - 1,$

(i) $-\infty < t < \infty$; *(ii)* $-1 \leq t \leq 1$.

(i)



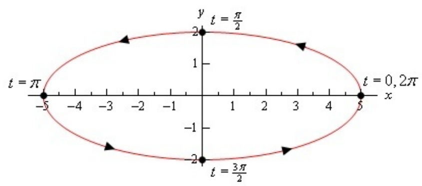
(ii)



Remark. If you eliminate the parameter t, then $x = \frac{1}{4}(y+1)^2 + \frac{1}{2}(y+1)$.

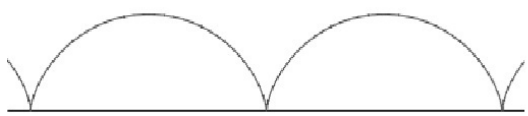
Example 19. *Sketch the parametric curve for the following set of parametric equations:*

$$x = 5\cos t, y = 2\sin t, 0 \leq t \leq 2\pi.$$



Example 20. *Sketch the Cycloid :*

$$x = r(t - \sin t), y = r(1 - \cos t), , -\infty < t < \infty.$$



How to parametrize curves?

- Parabola: $(x - x_0)^2 = 4p(y - y_0)$. Let $x = t$, then $y = \frac{1}{4p}(t - x_0)^2 + y_0$.
- Ellipse: $\dfrac{(x - x_0)^2}{a^2} + \dfrac{(y - y_0)^2}{b^2} = 1$. Let $x = x_0 + a\cos t$, $y = y_0 + b\sin t$.
- Hyperbola: $\dfrac{(x - x_0)^2}{a^2} - \dfrac{(y - y_0)^2}{b^2} = 1$. May use two ways:
  1. $x = x_0 + a\sec t$, $y = y_0 + b\tan t$, $-\pi < t < \pi$.
  2. $x = x_0 + a\cosh t$, $y = y_0 + b\sinh t$, $-\infty < t < \infty$, where

$$\cosh t = \frac{1}{2}(e^t + e^{-t}), \quad \sinh t = \frac{1}{2}(e^t - e^{-t}).$$

Example 21. *Parameterize the following curves:*

1. $3x + 2y^2 - 5y + 1 = 0$.

2. $4x^2 + 9y^2 - 8x + 36y + 4 = 0$.

3. $4x^2 - 9y^2 - 8x - 36y - 68 = 0$.

Solution:

1. $y = t, \quad x = -\dfrac{2}{3}t^2 + \dfrac{5}{3}t - \dfrac{1}{3}.$

2. $\dfrac{(x-1)^2}{9} + \dfrac{(y+2)^2}{4} = 1.$

3. $\dfrac{(x-1)^2}{9} - \dfrac{(y+2)^2}{4} = 1.$

The Calculus of Parametric Equations
Tangents

We want to find the tangent lines to the parametric equations given by, $x = f(t), y = g(t)$. By Chain Rule, we have

- First Derivative for Parametric Equations:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}, \ \textit{provided} \ \frac{dx}{dt} \neq 0.$$

- Second Derivative for Parametric Equations:

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{g''f' - f''g'}{f'^3}, \ \textit{provided} \ \frac{dx}{dt} \neq 0.$$

Example 22. *Find the tangent line(s) to the parametric curve given by*

$$x = t^5 - 4t^3, y = t^2, \ \textit{at} \ (0, 4).$$

Solution: At first we need the slope of the tangent line.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t}{5t^4 - 12t^2} = \frac{2}{5t^3 - 12t}.$$

When $(x, y) = (0, 4)$, $t = \pm 2$. At $t = -2$, the slope of the tangent line is: $-1/8$. The tangent line at $t = -2$ is: $y = 4 - x/8$; At $t = 2$, the slope is: $1/8$. The tangent line (at $t = 2$) is: $y = 4 + x/8$.

Example 23. *Find the points where the following parametric equations will have horizontal or vertical tangents:*

$$x = t^3 - 3t, y = 3t^2 - 9.$$

Solution: Horizontal Tangents: $dy/dt = 0, 6t = 0, t = 0$. Therefore, the only horizontal tangent will occur at the point $(x, y) = (0, -9)$.

Vertical Tangents: $dx/dt = 0$ (dy/dx undefined). In this case we need to solve, $3(t^2 - 1) = 0, \Rightarrow t = 1, -1$. The two vertical tangents will occur at the points (2,-6) and (-2,-6).

Example 24. *Determine the values of t for which the parametric curve given by the following set of parametric equations is concave up and concave down.*

$$x = 1 - t^2, y = t^7 + t^5.$$

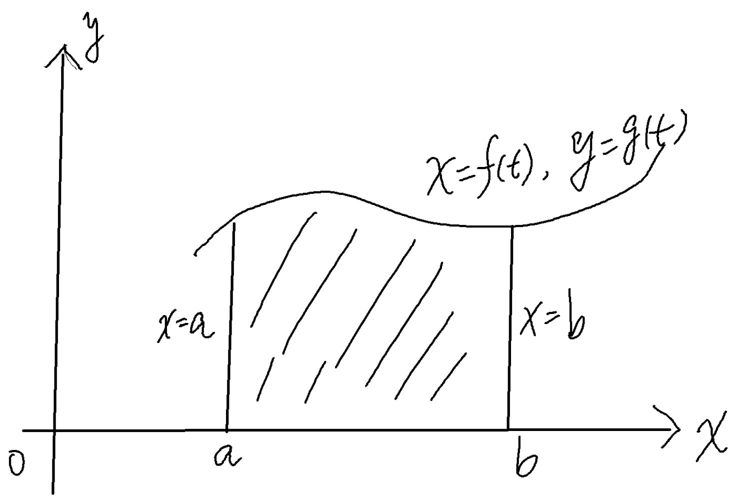
Solution: To study concavity, we need the second derivative.

$$\frac{dy}{dx} = -\frac{7t^5 + 5t^3}{2} \Rightarrow \frac{d^2y}{dx^2} = \frac{35t^3 + 15t}{4}.$$

From $\frac{d^2y}{dx^2} = 0$ we imply that t=0. When $t < 0$, $\frac{d^2y}{dx^2} < 0$, the parametric curve will be concave down; when $t > 0$, $\frac{d^2y}{dx^2} > 0$, the parametric curve will be concave up.

2.9 Applications to Area Problems

Consider the following region:



By calculus I, the area is:

$$A = \int_a^b y(x)\, dx.$$

Theorem 1. *The area under the parametric curve $x = f(t), y = g(t)$ between $\alpha \le t \le \beta$ is*

$$A = \int_\alpha^\beta y(t)x'(t)dt, \;\; \textit{where} \;\; x(\alpha) = a, x(\beta) = b.$$

Example 25. *Find the area under one arch of cycloid:*

$$x = r(t - \sin t), y = r(1 - \cos t).$$

Solution: $y = 0 \Rightarrow t = 2n\pi$. Thus

$$A = \int_\alpha^\beta y(t)x'(t)dt = \int_0^{2\pi} r^2(1-\cos t)^2 dt = r^2 \int_0^{2\pi} (1 - 2\cos t + \frac{1+\cos 2t}{2})dt = 3\pi r^2.$$

Theorem 2. *For the closed curve $C: x = f(t), y = g(t), \alpha \leq t \leq \beta, f(\alpha) = f(\beta), g(\alpha) = g(\beta)$, the area of the region R enclosed by C is*

$$A(R) = |\int g(t)f'(t)dt|.$$

Example 26. *Find the area of the ellipse $x = a\cos t, y = b\sin t$, $0 \leq t \leq 2\pi$, $a > 0, b > 0$.*

Solution:

$$A = |\int_0^{2\pi} -ab\sin^2 t dt| = | -\frac{1}{2}ab \int_0^{2\pi} (1 - 2\cos 2t)dt| = \pi ab.$$

2.10 Arc Length

Here we will find a formula for determining the arc length to a parametric curve given by the parametric equations:

$$x = f(t), y = g(t), \alpha \leq t \leq \beta.$$

We assume that the curve is traced out exactly once as t increases from α to β. Also, for the purposes of the derivation that we're going to use, we will assume that the curve is traced out from left to right as t increases. This is equivalent to saying,

$$dx/dt \geq 0, \alpha \leq t \leq \beta.$$

The arc length formula is given by

$$L = \int_a^b \sqrt{1 + [y'(x)]^2}\, dx = \int_\alpha^\beta \sqrt{[x'(t)]^2 + [y'(t)]^2} dt, \ \ where\ \ x(\alpha) = a, x(\beta) = b.$$

Example 27. *Find the length under one arch of cycloid:*

$$x = r(t - \sin t), y = r(1 - \cos t),\ 0 \leq t \leq 2\pi.$$

Solution:

$$L = \int_{\alpha}^{\beta} \sqrt{[x'(t)]^2 + [y'(t)]^2} dt = \int_0^{2\pi} \sqrt{[r(1-\cos t)]^2 + [r\sin t]^2} dt$$

$$= r\int_0^{2\pi} \sqrt{2 - 2\cos t} dt = r\int_0^{2\pi} 2\left|\sin\frac{t}{2}\right| dt = 8r.$$

Example 28. *Find the length of the curve:*

$$C: \ x = e^t + e^{-t}, y = 5 - 2t, \ 0 \leq t \leq 3.$$

Solution:

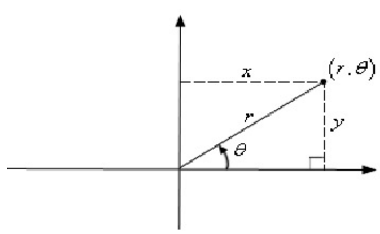
$$L = \int_{\alpha}^{\beta} \sqrt{[x'(t)]^2 + [y'(t)]^2} dt = \int_0^{3} \sqrt{(e^t - e^{-t})^2 + (-2)^2} dt$$

$$= \int_0^3 \sqrt{(e^t + e^{-t})^2} dt = \int_0^3 (e^t + e^{-t}) dt = e^3 - e^{-3}.$$

2.11-2.14. Polar Coordinates

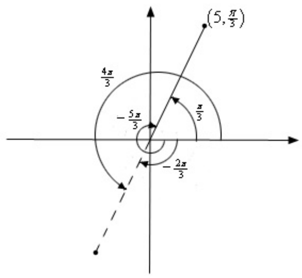
Polar coordinates

We chose a point in the plane that is called the pole (or origin) and is labelled O. Then we draw a ray starting at O, along positive x-axis, which is called the polar axis. Let P be a point in the plane. Let r be the distance from P to O, let θ be the angle between OP and the polar axis. Then P can be represented by the ordered pair (r, θ). We call r and θ polar coordinates:



Agreement: $(-r, \theta) = (r, \theta + \pi)$.

Example 29. *Sketch of several polar coordinates:*



Polar $\rightleftarrows$ Cartesian Conversion Formulas:

$$x = r\cos\theta, y = r\sin\theta; \quad r = \sqrt{x^2 + y^2}, \tan\theta = \frac{y}{x}.$$

Example 30. *Convert each of the following points into the given coordinate system.*
(a) $(-4, \frac{2\pi}{3})$ *into Cartesian coordinates. (b) (-1,-1) into polar coordinates.*

Solution: (a) $(x, y) = (2, -2\sqrt{3})$.
(b) $r = \sqrt{x^2 + y^2} = \sqrt{2}, \tan\theta = \frac{y}{x} = 1$. Since the point is in the third quadrant, the actual angle is, $\theta = \frac{\pi}{4} + \pi = \frac{5\pi}{4}$.

Polar curves: $r = f(\theta)$

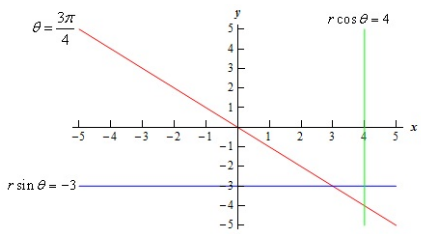
The graph of polar equation $r = f(\theta)$, or more generally $F(r, \theta) = 0$, consists of all points $P(r, \theta)$ whose coordinates satisfy the equation. Relation to Cartesian:

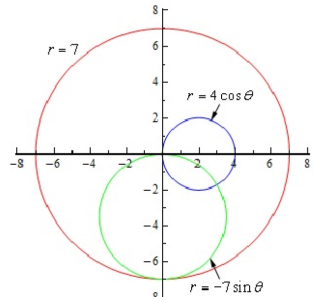
$$x = r\cos\theta = f(\theta)\cos\theta, \quad y = r\sin\theta = f(\theta)\sin\theta.$$

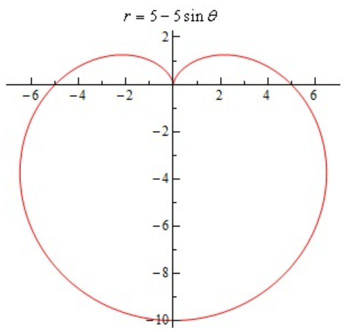
Some special cases:

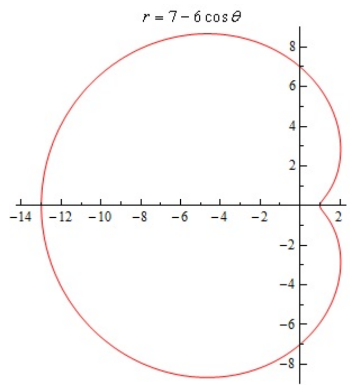
- $\theta = \alpha$: This is a line that goes through the origin and makes an angle of α with the positive x-axis.
- $r\cos\theta = a$: This is equivalent to $x = a$.
- $r\sin\theta = b$: This is equivalent to $y = b$.
- $r = a$: A circle of radius a centered at the origin.
- $r = 2a\cos\theta$: A circle of radius $|a|$ and center $(a, 0)$.
- $r = 2b\sin\theta$: A circle of radius $|b|$ and center $(0, b)$.
- $r = 2a\cos\theta + 2b\sin\theta$: A circle of radius $r = \sqrt{a^2 + b^2}$ and center (a, b).

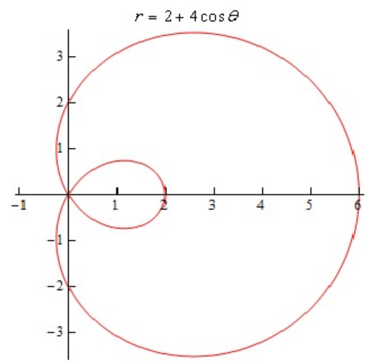
Example 31.











Remark. In the third graph we have an inner loop. To get this, we need to know the value of θ for which the graph will pass through the origin:

$$2 + 4\cos\theta = 0, \Rightarrow \cos\theta = -0.5, \Rightarrow \theta = 2\pi/3, 4\pi/3.$$

Tangents to Polar Curves

We want to find the tangent lines to the equation $r = f(\theta)$. From

$$x = r\cos\theta = f(\theta)\cos\theta, y = r\sin\theta = f(\theta)\sin\theta$$

we have

$$\frac{dy}{dx} = \frac{r'_\theta \sin\theta + r\cos\theta}{r'_\theta \cos\theta - r\sin\theta}.$$

Example 32. *Find the equation of the tangent line to*

$$r = 3 + 8\sin\theta \ \textit{at} \ \theta = \pi/6.$$

Solution:

$$\frac{dy}{dx} = \frac{r'_\theta \sin\theta + r\cos\theta}{r'_\theta \cos\theta - r\sin\theta} = \frac{16\cos\theta\sin\theta + 3\cos\theta}{8\cos^2\theta - 3\sin\theta - 8\sin^2\theta} = \frac{11\sqrt{3}}{5}.$$

Note that at $\theta = \pi/6, r = 7$, which gives $(x, y) = (\frac{7\sqrt{3}}{2}, \frac{7}{2})$. The tangent line is:

$$y = \frac{11\sqrt{3}}{5}x - \frac{98}{5}.$$

Example 33. *Cardioid:* $r = 1+\sin\theta$ *. Find* θ *where we have horizontal or vertical tangent line, or no tangent line.*

Solution:
$$\frac{dy}{dx} = \frac{r'_\theta \sin\theta + r\cos\theta}{r'_\theta \cos\theta - r\sin\theta} = \frac{\cos\theta(1+2\sin\theta)}{(1+\sin\theta)(1-2\sin\theta)}.$$

(1) If the denominator is 0 but the numerator is not 0, we have vertical tangent line. When $sin\theta = 1$, or$sin\theta = 1/2$, the denominator is 0. We have a vertical tangent line when $\theta = \pi/6, 5\pi/6$.

(2) If the denominator is not 0 but the numerator is 0, we have a horizontal tangent line. Hence, we have a horizontal tangent line when $= \pi/2, 7\pi/6, 11\pi/6$.

(3) If the denominator is 0 and the numerator is 0, tangent line does not exist. When $\theta = 3\pi/2$, the tangent line does not exist.

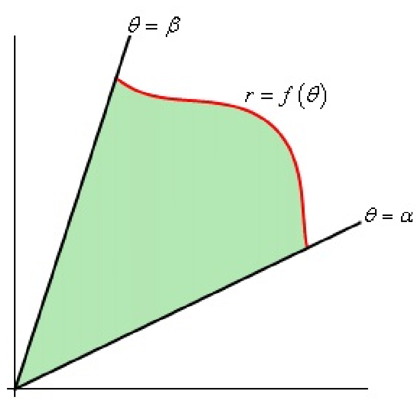
Area

As we know, the area of a sector with radius r and angle θ is

$$A = \pi r^2 \frac{\theta}{2\pi} = \frac{1}{2} r^2 \theta.$$

This implies that the area of the following polar region bounded by $r = f(\theta)$, between $\theta = \alpha$ and $\theta = \beta$ $(\alpha \leq \beta)$ is:

$$A = \int_\alpha^\beta \frac{1}{2} r^2 d\theta.$$



Example 34. *Find the area of the inner loop of* $r = 2 + 4\cos\theta$*.*

Solution: Let $r = 0$ we have:

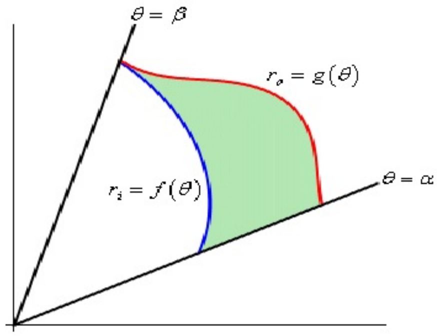
$$2 + 4\cos\theta = 0, \Rightarrow cos\theta = -0.5, \Rightarrow \theta = 2\pi/3, 4\pi/3.$$

So the inner loop is bounded by $r = 2 + 4\cos\theta$ and between $\theta = 2\pi/3$ and $4\pi/3$. Thus

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 d\theta = \int_{2\pi/3}^{4\pi/3} \frac{1}{2}(2 + 4\cos\theta)^2 d\theta.$$

$$= \int_{2\pi/3}^{4\pi/3} [6 + 8\cos\theta + 4\cos(2\theta)]d\theta = 4\pi - 6\sqrt{3}.$$

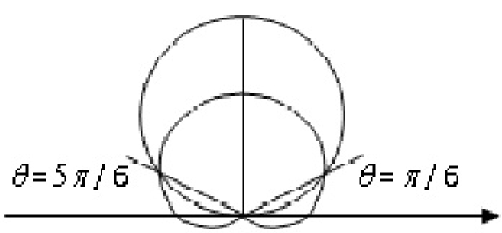
Now we consider a more general polar region:



The area of the shaded part will be :

$$A = \int_{\alpha}^{\beta} \frac{1}{2}(r_0^2 - r_i^2) d\theta.$$

Example 35. *Find the area of the part outside the cardioid $r = 1 + \sin\theta$, inside the circle $r = 3\sin\theta$.*



Solution: First find intersections: $1 + \sin\theta = 3\sin\theta \Rightarrow \sin\theta = 1/2 \Rightarrow \theta = \pi/6, 5\pi/6$. When $\pi/6 \leq \theta \leq 5\pi/6$, $3\sin\theta \geq 1 + \sin\theta$.

$$A = \int_{\alpha}^{\beta} \frac{1}{2}(r_0^2 - r_i^2)d\theta = \int_{\pi/6}^{5\pi/6} \frac{1}{2}[(3\sin\theta)^2 - (1 + \sin\theta)^2]d\theta = \pi.$$

Arc length

Now we are going to find the formula for the arc length of the arc $r = f(\theta)$, between $\theta = \alpha$ and $\theta = \beta$ $(\alpha \leq \beta)$. Note that

$$x = r\cos\theta = f(\theta)\cos\theta, y = r\sin\theta = f(\theta)\sin\theta, \Rightarrow$$

$$(x'_\theta)^2 + (y'_\theta)^2 = (r'_\theta)^2 + r^2.$$

Thus

$$L = \int_\alpha^\beta \sqrt{(x'_\theta)^2 + (y'_\theta)^2} d\theta = \int_\alpha^\beta \sqrt{(r'_\theta)^2 + r^2} d\theta.$$

Example 36. *Find the length of the spiral* $r = \theta$, $0 \le \theta \le 1$.

Solution:

$$\begin{aligned} L &= \int_\alpha^\beta \sqrt{(r'_\theta)^2 + r^2} d\theta = \int_0^1 \sqrt{\theta^2 + 1} d\theta \\ &= \int_0^{\pi/4} \sec^3 x dx, \quad \theta = \tan x, d\theta = \sec^2 x dx \\ &= (\sec x \tan x + \ln|\sec x + \tan x|)|_0^{\pi/4} = \frac{1}{2}(\sqrt{2} + \ln(1+\sqrt{2}). \end{aligned}$$

Example 37. *Find the length of the cardioid* $r = 1 + \sin\theta$, $0 \le \theta \le 2\pi$.

Solution:

$$\begin{aligned} L &= \int_\alpha^\beta \sqrt{(r'_\theta)^2 + r^2} d\theta = \int_0^{2\pi} \sqrt{2(\sin\theta + 1)} d\theta \\ &= \sqrt{2} \int_0^{2\pi} |\sin\frac{\theta}{2} + \cos\frac{\theta}{2}| d\theta = 2\sqrt{2} \int_0^{\pi} |\sin\varphi + \cos\varphi| d\varphi, \quad \varphi = \theta/2 \\ &= 2\sqrt{2} \int_0^{3\pi/4} (\sin\varphi + \cos\varphi) d\varphi - 2\sqrt{2} \int_{3\pi/4}^{\pi} (\sin\varphi + \cos\varphi) d\varphi \\ &= 8. \end{aligned}$$

Chapter 3: Vector Calculus

3.1 Continuity

- A function of two variables $z = f(x, y)$ is a rule which maps each point (x, y) in a set D to a unique number z. The set D is called the domain of the function, which is often denoted $D(f)$. Level curves (contour maps) of $f(x, y)$: $f(x, y) = k$ for different k.

- A function of three variables $w = f(x, y, z)$ is a rule which maps each point (x, y, z) in a set D to a unique number w. The set D is called the domain of the function, which is often denoted $D(f)$. Level surfaces of $f(x, y, z)$: $f(x, y, z) = k$ for different k.

Limit and continuity.

- If $f(x, y)$ can be made as close to L as (x, y) close to (a, b), then

$$\lim_{(x,y)\to(a,b)} f(x, y) = L.$$

  In general, if there are different limits when (x, y) approaches (a, b) along different paths, then the limit does not exist. $f(x, y)$ is continuous at (a, b) if

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b).$$

- If $f(x, y, z)$ can be made as close to L as (x, y, z) close to (a, b, c), then

$$\lim_{(x,y,z)\to(a,b,c)} f(x, y, z) = L.$$

Example 38. *Assume that the limit* $\lim_{(x,y)\to(0,0)} \frac{4x^2y}{x^2 + y^2}$ *exists. Find it.*

Solution: By taking the path $y = x$, then $\frac{4x^2y}{x^2 + y^2} = 2x \to 0$.

Example 39. *Show that the limit* $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2}$ *does not exist. Show that*

$$f(x) = \begin{cases} \frac{xy}{x^2+y^2} & if\ (x, y) \neq (0, 0) \\ 0 & if\ (x, y) = (0, 0) \end{cases}$$

is discontinuous at $(0, 0)$.

Solution: Since the limit is $1/2$ when $y = x$; and the limit is 0 when $y = 0$.

Example 40.

$$\lim_{(x,y,z)\to(1,2,3)} \frac{xy + yz + xz}{xyz - 1} = \frac{11}{5}.$$

3.2-3.3 Partial Derivatives

Functions of two variables

- Partial derivatives of $z = f(x, y)$:

$$z_x = \frac{\partial z}{\partial x} := \frac{\partial f}{\partial x} := f_x(x, y) := D_x f := \lim_{h\to 0} \frac{f(x + h, y) - f(x, y)}{h},$$

  which is the derivative of f with respect to x;

$$z_y = \frac{\partial z}{\partial y} := \frac{\partial f}{\partial y} := f_y(x, y) := D_y f := \lim_{h\to 0} \frac{f(x, y + h) - f(x, y)}{h},$$

  which is the derivative of f with respect to y.

- Methods:
  1. To find f_x: regard y as a constant, and differentiate $f(x, y)$ with respect to x;
  2. To find f_y: regard x as a constant, and differentiate $f(x, y)$ with respect to y.

- Meaning: f_x means the rate of change of f with respect to x when y is fixed.

Example 41. *Let*

$$f(x) = \begin{cases} \frac{xy}{x^2+y^2} & if \ (x, y) \neq (0, 0) \\ 0 & if \ (x, y) = (0, 0) \end{cases}$$

(a) Using the definition of a partial derivative (do not differentiate) find $f_x(0, 0)$.

(b) Using the definition of a partial derivative (do not differentiate) find $f_x(0, 0)$.

Solution:

(a)

$$\frac{\partial f}{\partial x}(0, 0) = \lim_{h\to 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h\to 0} \frac{0 - 0}{h} = 0.$$

(b)

$$\frac{\partial f}{\partial y}(0, 0) = \lim_{k\to 0} \frac{f(0, k) - f(0, 0)}{k} = \lim_{k\to 0} \frac{0 - 0}{k} = 0.$$

Remark. This example shows that the existence of partial derivatives at a point is insufficient to guarantee that the function is continuous there.

Example 42. *Let* $f(x,y) = e^{xy} + \frac{x}{y}$. *Calculate* $f_x(0,1)$, $f_y(0,1)$.

Solution:

$$f_x = ye^{xy} + \frac{1}{y}, \quad f_x(0,1) = 2.$$
$$f_y = xe^{xy} - \frac{x}{y^2}, \quad f_y(0,1) = 0.$$

Functions of three variables

- Let $w = f(x,y,z)$, then
$$\frac{\partial w}{\partial x} := \frac{\partial f}{\partial x} := f_x(x,y,z) := D_x f := \lim_{h\to 0} \frac{f(x+h,y,z) - f(x,y,z)}{h},$$
which is the derivative of f with respect to x.

- Meaning: f_x means the rate of change of f (or w) with respect to x when y and z are fixed.

Example 43. *Let* $f(x,y,z) = (\sin z)e^{xy} \ln x$. *Calculate* f_x, f_y, f_z.

Solution:

$$f_x = (\sin z)e^{xy}\frac{1}{x}, \quad f_y = (\sin z)xe^{xy} \ln x, \quad f_z = (\cos z)e^{xy} \ln x.$$

Implicit differentiation:

Example 44. *Find* $\frac{\partial z}{\partial x}$ *and* $\frac{\partial z}{\partial y}$, *if* z *is implicitly defined by*

$$x^2 + y^3 + z^4 - 8xyz = 1.$$

Solution: Differentiate two sides with respect to x:

$$2x + 4z^3 z_x - 8yz - 8xyz_x = 0, \quad z_x = \frac{-2x + 8yz}{4z^3 - 8xy}.$$

Higher derivatives:

f_{xx}, $\frac{\partial^3 f}{\partial z \partial y \partial x} = f_{xyz}, \ldots$

Example 45. *Let* $f(x,y) = e^{xy} + \frac{x}{y}$. *Calculate* f_{xx}, f_{xy}, f_{yy}.

Solution:

$$f_x = ye^{xy} + \frac{1}{y}, f_y = xe^{xy} - \frac{x}{y^2}.$$

$$f_{xx} = y^2e^{xy}, \quad f_{xy} = e^{xy} + xye^{xy} - \frac{1}{y^2}, \quad f_{yy} = x^2e^{xy} + \frac{2x}{y^3}.$$

Example 46. *Let* $f(x,y,z) = \sin ze^{xy} \ln x$. *Calculate* f_{xyz}.

Solution: $f_x(x,y,z) = y \sin ze^{xy} \ln x + \sin ze^{xy}/x$.
$f_{xy}(x,y,z) = \sin ze^{xy} \ln x + y^2 \sin ze^{xy} \ln x + \sin ze^{xy}$.
$f_{xyz}(x,y,z) = \cos ze^{xy} \ln x + y^2 \cos ze^{xy} \ln x + \cos ze^{xy}$.

Clairaut's Theorem. If $f(x,y)$ is defined in a disk D containing (a,b), and f_{xy} and f_{yx} are both continuous in D, then $f_{xy}(a,b) = f_{yx}(a,b)$.

3.5 Directional Derivatives and Gradients

Gradients and Directional Derivatives in the Plane

- Directional derivatives:

  1. The directional derivative of the function $f(x,y)$ at (x_0,y_0) in the direction of a unit vector $\vec{u} = (u_1,u_2)$ is

$$\begin{aligned} D_{\vec{u}}f(x_0,y_0) &= \lim_{h\to 0} \frac{f(x_0+hu_1, y_0+hu_2) - f(x_0,y_0)}{h} \\ &= f_x(x_0,y_0)u_1 + f_y(x_0,y_0)u_2 \\ &= (f_x(x_0,y_0), f_y(x_0,y_0)) \cdot (u_1,u_2). \end{aligned}$$

  2. $D_{\vec{u}}f(x_0,y_0)$ means the rate of change of $f(x,y)$ at (x_0,y_0) in the direction of $\vec{u}$.

- The gradient of $f(x,y)$ at (x_0,y_0) is

$$\nabla f(x_0,y_0) = gradf(x_0,y_0) = (f_x(x_0,y_0), f_y(x_0,y_0)).$$

  1. $\nabla f(x_0,y_0)$ points into the direction of maximum increase of f at (x_0,y_0).

2. $\nabla f(x_0, y_0)$ is perpendicular to the contour line (or level curve) of f through (x_0, y_0).

3. $|\nabla f(x_0, y_0)|$ is the maximum rate of change of f at (x_0, y_0).

Example 47. *Let* $f(x, y) = x^2y + 4y^2$.

(1) Calculate the gradient of the function.

(2) Find the directional derivative of $f(x, y)$ *at the point* $(2, 1)$ *in the direction of the vector* $\langle 1, \sqrt{3}\,\rangle$.

(3) Find the maximum rate of change of f at (2, 1) and indicate in which direction this maximum will occur.

Solution: (1) $\nabla f = \langle 2xy, x^2 + 8y\rangle$.
(2) $\vec{v} = \langle 1, \sqrt{3}\,\rangle$, $\vec{u} = \frac{1}{2}\langle 1, \sqrt{3}\,\rangle$,, $\nabla f(2, 1) = \langle 4, 12\rangle$.
$D_{\vec{u}} f = \nabla f \cdot \vec{u}$
$D_{\text{u}} f(2, 1) = \nabla f(2, 1) \cdot \vec{u} = \langle 4, 12\rangle \cdot \frac{1}{2}\langle 1, \sqrt{3}\,\rangle = 2 + 6\sqrt{3}$
(3) The maximum rate of change of f at (2, 1) $= |\nabla f(2, 1)| = |\langle 4, 12\rangle| = 4\sqrt{10}$, which occurs in the direction $\langle 4, 12\rangle$.

Gradient and Directional Derivatives in space

- Directional derivatives:

  1. The directional derivative of the function $f(x, y, z)$ at (x_0, y_0, z_0) in the direction of a unit vector $\vec{u} = (u_1, u_2, u_3)$ is

$$f_{\vec{u}}(x_0, y_0, z_0) = D_{\vec{u}} f(x_0, y_0, z_0) = f_x(x_0, y_0, z_0)u_1 + f_y(x_0, y_0, z_0)u_2 + f_z(x_0, y_0, z_0)u_3.$$

- The gradient of $f(x, y, z)$ at (x_0, y_0, z_0) is

$$gradf(x_0, y_0, z_0) = \nabla f(x_0, y_0, z_0) = (f_x(x_0, y_0, z_0), f_y(x_0, y_0, z_0), f_z(x_0, y_0, z_0)).$$

  1. $\nabla f(x_0, y_0, z_0)$ points into the direction of maximum increase of f at (x_0, y_0, z_0).
  2. $\nabla f(x_0, y_0, z_0)$ is perpendicular to the level surface of f through (x_0, y_0, z_0).
  3. $|\nabla f(x_0, y_0, z_0)|$ is the maximum rate of change of f at (x_0, y_0, z_0).

Example 48. *Suppose that the temperature of a room at a point* (x, y, z) *is given by*

$$T(x, y, z) = \frac{80}{1 + x^2 + 2y^2 + 3z^2}\ C^o.$$

(a) In which direction does the temperature increase fastest at the point $(2, 1, 1)$*?*

(b) What is the maximum rate of increase?

(c) Find the directional derivative of the function $T(x, y, z)$ at $(2, 1, 1)$ in the direction $(2, -1, -2)$.

Solution: (a)

$$\nabla T(x, y, z) = T_x i + T_y j T_z k = \frac{160}{(1 + x^2 + 2y^2 + 3z^2)^2}(-xi - 2yj - 3zk),$$

$$\nabla T(2, 1, 1) = \frac{8}{5}(-2i - 2j - 3k).$$

(b)

$$|\nabla T(2, 1, 1)| = \frac{8\sqrt{17}}{5}.$$

(c) $\vec{u} = \frac{(2,-1,-2)}{|(2,-1,-2)|} = (\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3})$.

$$T_{\vec{u}}(2, 1, 1) = D_{\vec{u}} f(2, 1, 1) = \frac{8}{5}(-2, -2, -3) \cdot (\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3}) = -\frac{32}{15}.$$

3.6 The Chain Rule

- Basic Chain Rule

  1. If $z = f(x, y)$, $x = g(t)$, $y = h(t)$, then
  $$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$
  2. $\frac{dz}{dt}$ means rate of change of z with respect to t along the path $x = g(t)$, $y = h(t)$, $t \in D$.
  3. If $w = f(x, y, z)$, $x = g(t)$, $y = h(t)$, $z = k(t)$, then
  $$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}.$$

- General Chain Rule: If $w = f(x, y, z)$, $x = g(u, v)$, $y = h(u, v)$, $z = k(u, v)$, then
$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u},$$
$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}.$$

- Implicit Differentiation:

1. If $F(x,y)=0$, then
$$\frac{dy}{dx}=-\frac{F_x}{F_y}.$$
Here when we calculate partial derivatives, we consider x and y as independent variables.

2. If $F(x,y,z)=0$, then
$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z},\quad \frac{\partial z}{\partial y}=-\frac{F_y}{F_z}.$$
Here when we calculate partial derivatives, we consider x, y and z as independent variables.

Example 49. *Let $z=f(x,y)$, where $x=g(t)$ and $y=h(t)$. Given the data $g(1)=1,\ g'(1)=2, h(1)=2, h'(1)=3, f_x(1,2)=-1,\ f_y(1,2)=2$. Find $\frac{dz}{dt}$ when $t=1$.*

Solution:
$$\frac{dz}{dt}=\frac{\partial z}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt}=f_xg'+f_yh'.$$
$$t=1\Rightarrow (x,y)=(g(1),h(1))=(1,2),\Rightarrow$$
$$\frac{dz}{dt}|_{t=1}=f_x(1,2)g'(1)+f_y(1,2)h'(1)=(-1)(2)+(2)(3)=4.$$

Example 50. *Consider the following function*
$$z=x^2y+e^x\cos y,\quad x=t^3\sin s,\ y=s^2+3t^2.$$
Calculate $\frac{\partial z}{\partial s}$ at the point $(s,t)=(0,1)$ by using Chain Rule.

Solution: At the point $(s,t)=(0,1)$, $(x,y)=(0,3)$. Note that $z_x=2xy+e^x\cos y$, $z_y=x^2-e^x\sin y,\ x_s=t^3\cos s,\ y_s=2s$. Thus
$$z_x(0,3)=\cos 3, z_y(0,3)=-\sin 3, x_s(0,1)=1, y_s(0,1)=0.$$
$$\frac{\partial z}{\partial s}|_{(s,t)=(0,1)}=\frac{\partial z}{\partial x}|_{(x,y)=(0,3)}\frac{\partial x}{\partial s}|_{(s,t)=(0,1)}+\frac{\partial z}{\partial y}|_{(x,y)=(0,3)}\frac{\partial y}{\partial s}|_{(s,t)=(0,1)}$$
$$=(\cos 3)1+(-\sin 3)0=\cos 3.$$

Example 51. *Find y' if $x^2y+e^x\cos y=3$.*

Solution: Let $f(x,y)=x^2y+e^x\cos y-3$. Then
$$f_x=2xy+e^x\cos y, f_y=x^2-e^x\sin y$$
Thus
$$y'=-\frac{f_x}{f_y}=-\frac{2xy+e^x\cos y}{x^2-e^x\sin y}.$$

Example 52. *Find* $\frac{\partial z}{\partial x}$ *and* $\frac{\partial z}{\partial y}$ *if* $x^2y^3 + z^4 + 5xyz = 3$.

Solution: Let $f(x, y, z) = x^2y^3 + z^4 + 5xyz - 3$. Then

$$f_x = 2xy^3 + 5yz, f_z = 4z^3 + 5xy$$

Thus

$$z_x = -\frac{f_x}{f_z} = -\frac{2xy^3 + 5yz}{4z^3 + 5xy}.$$

3.6.3 Tangent plane and Normal lines.

Curves in 3D:

$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k} = (x(t), y(t), z(t)), \quad a \le t \le b.$$

The curve is regular curve: if $\vec{r}\,'(t) \neq 0$ for all t.

- Tangent vector: $\vec{r}\,'(t) = x'(t)\vec{i} + y'(t)\vec{j} + z'(t)\vec{k} = (x'(t), y'(t), z'(t))$.
- Tangent line to a curve at a point t_0: The line at t_0 with $\vec{r}\,'(t_0)$ as the direction vector.
- Normal plane to a curve $\vec{r}(t) = (x(t), y(t), z(t))$ at the point $P(a, b, c)$, where $a = x(t_0)$, $b = y(t_0)$, $c = z(t_0)$, is:

$$x'(t_0)(x - a) + y'(t_0)(y - b) + z'(t_0)(z - c) = 0.$$

Example 53. *Let* $C: \ \vec{r}(t) = (t, 2\sin t, 2\cos t), \quad 0 \le t \le \pi$. *Find the tangent line and normal plane to the curve at* $t = \pi/3$.

Solution: (a) Tangent line:

$$\vec{r}\,'(t) = (1, 2\cos t, -2\sin t).$$

$$\vec{r}(\pi/3) = (\pi/3, \sqrt{3}, 1), \quad \vec{r}\,'(\pi/3) = (1, 1, -\sqrt{3}).$$

The tangent line is:

$$(x, y, z) = (\pi/3, \sqrt{3}, 1) + t(1, 1, -\sqrt{3}).$$

(b) The normal plane is

$$(1, 1, -\sqrt{3}) \cdot (x - \pi/3, y - \sqrt{3}, z - 1) = 0, i.e., x + y - \sqrt{3}z - \pi/3 = 0.$$

Example 54. *Find the equation of the tangent line and the normal plane at the point* $P(1,-1,3)$ *to the curve of the intersection of the surfaces* $2x^2+3y^2=5$ *and* $y^2+z^2=10$.

Solution: $\vec{r}(t)=(x(t),y(t),z(t))=(t,-\sqrt{\frac{5-2t^2}{3}},\sqrt{\frac{25+2t^2}{3}})$.
At P, $t=1$.
Tangent line is:

$$\frac{x-1}{1}=\frac{y+1}{2/3}=\frac{z-3}{2/9}.$$

Normal plane is:

$$1(x-1)+\frac{2}{3}(y+1)+\frac{2}{9}(z-3)=0,\Rightarrow 9x+6y-4z=3.$$

Remark. We have some other ways to parameterize the curve.

Surfaces in 3D: The surface S given by $F(x,y,z)=0$.

- The tangent plane of the surface S given by $F(x,y,z)=0$ at $P(a,b,c)$ is the plane that passes through P and has normal vector $\nabla F(a,b,c)$. Thus the equation is

$$(F_x(a,b,c),F_y(a,b,c),F_z(a,b,c))\cdot(x-a,y-b,z-c)=0.$$

- The normal line to the surface S given by $F(x,y,z)=0$ at $P(a,b,c)$ is the line that passes through P and has the direction vector $\nabla F(a,b,c)$. The equation is:

$$\frac{x-a}{F_x(a,b,c)}=\frac{y-b}{F_y(a,b,c)}=\frac{z-c}{F_z(a,b,c)}.$$

- Equation of the tangent plane of $z=f(x,y)$ at (x_0,y_0):

$$z=f(x_0,y_0)+f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0).$$

- Linear Approximation (Tangent plane approximation):

$$\begin{aligned} f(x,y) &\approx f(x_0,y_0)+f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)\\ &= L(x,y). \end{aligned}$$

Example 55. *Find the equation of the tangent plane of the surface* $z=e^{x+y}-\frac{x}{y}$ *at the point* $(1,-1,2)$.

Solution: Let $f(x,y) = e^{x+y} - \frac{x}{y}$. Then

$$f_x = e^{x+y} - \frac{1}{y}, \quad f_x(1,-1) = 2.$$

$$f_y = e^{x+y} + \frac{x}{y^2}, \quad f_y(1,-1) = 2.$$

Thus the equation of the tangent plane at the point $(1,-1,2)$ is

$$z - 2 = 2(x-1) + 2(y+1), \quad i.e., \quad 2x + 2y - z + 2 = 0.$$

Example 56. *Find the equation of the tangent plane and the normal line at the point (2,1,9) to the ellipsoid* $\frac{x^2}{12} + \frac{y^2}{3} + \frac{z^2}{27} = \frac{11}{3}$.

Solution: Let $F(x,y,z) = \frac{x^2}{12} + \frac{y^2}{3} + \frac{z^2}{27} - \frac{11}{3}$. Then the normal vector of the tangent plane is

$$\vec{n} = (F_x(2,1,9), F_y(2,1,9), F_z(2,1,9)) = (1/3, 2/3, 2/3).$$

Tangent plane: $x + 2y + 2z - 22 = 0$.

The normal line is: $(x,y,z) = (2,1,9) + t(1/3, 2/3, 2/3)$.

Example 57. *Use the linear approximation of* $f(x,y) = 2x^2y^2 + 3xy + x$ *at* $(1,1)$ *to approximate* $f(0.9, 1.1)$.

Solution: $f(x,y) \approx f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1)$.
$f(x,y) = 2x^2y^2 + 3xy + x,\ f_x = 4xy^2 + 3y + 1,\ f_y = 4x^2y + 3y$
$f(1,1) = 6,\ f_x(1,1) = 8,\ f_y(1,1) = 7$
Thus $f(x,y) \approx 6 + 8(x-1) + 7(y-1)$.
$f(0.9,\ 1.1) \approx 6 + 8 * (-0.1) + 7 * 0.1 = 5.9$

3.7 Conservative Fields

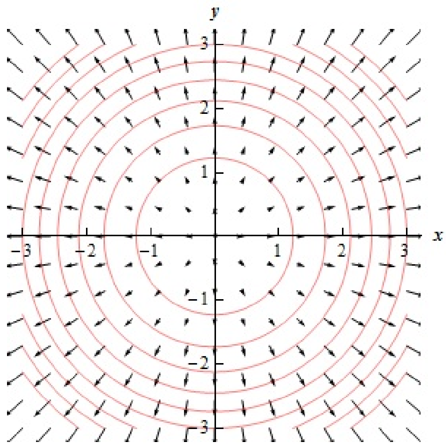
Vector Fields

In a two-dimensional space,vector function $\vec{F}(x,y) = (P(x,y), Q(x,y)) = P(x,y)\vec{i} + Q(x,y)\vec{j}$ is a 2-dimensional vector field.

In a three-dimensional space, $\vec{F}(x,y,z) = (P(x,y,z), Q(x,y,z), R(x,y,z)) = P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k}$ is a 3-dimensional vector field.

Vector fields can be visualized by diagrams.

Example 58. *Gradient Vector Fields: Let* $f(x,y) = x^2 + y^2$*. Then* $\vec{F}(x,y) = \nabla f(x,y) = (x,y)$ *is a vector field.*



Conservative Vector Fields

A vector field $\vec{F}$ is conservative if there exists a function f such that $\vec{F} = \nabla f$. In other words, a vector field is conservative if it is the gradient field of a (scalar) function. The function f is called a **potential function** of $\vec{F}$.
Remark. The potential function of a conservative vector field is not unique.

Example 59. *(a) Verify that* $\vec{F} = (y\cos x, \sin x)$ *is a conservative vector field with potential function* $f(x,y) = y\sin x$.

(b) Verify that $\vec{F} = (y^2, 2xy + e^{3z}, 3ye^{3z})$ *is a conservative vector field with potential function* $f(x,y,z) = xy^2 + ye^{3z}$.

Solution: (a) $\nabla f = (f_x, f_y) = (y\cos x, \sin x)$. Thus $\vec{F} = \nabla f$.
(b) $\nabla f = (f_x, f_y, f_z) = (y^2, 2xy + e^{3z}, 3ye^{3z})$. Thus $\vec{F} = \nabla f$.

A Necessary and Sufficient Condition for a Vector Field to be Conservative:

Let $\vec{F} = (P, Q)$ be a vector field in a simply-connected region D. Suppose P and Q have continuous first-order derivative in D, then $\vec{F}$ is conservative if and only if

$$P_y = Q_x.$$

Example 60. *(i) Show that the vector field* $\vec{F} = (3 + 2xy, x^2 - 3y^2)$ *is conservative.*

(ii) Find a potential function of this field.

Solution: (i) Let $P = 3 + 2xy$, $Q = x^2 - 3y^2$. Since $P_y = 2x = Q_x$, $\vec{F}$ is conservative.

(ii) Let $f(x,y)$ be a potential function. Then $F = (f_x, f_y) = (P, Q)$,

$$f_x = 3 + 2xy, f_y = x^2 - 3y^2.$$

$$f_x = 3 + 2xy \Rightarrow f(x,y) = 3x + x^2y + g(y) \Rightarrow f_y = x^2 + g'(y) \Rightarrow$$
$$x^2 - 3y^2 = x^2 + g'(y), \Rightarrow g'(y) = -3y^2 \Rightarrow g(y) = -y^3.$$

Hence,

$$f(x,y) = 3x + x^2y - y^3.$$

Example 61. *Find a potential function $f(x,y,z)$ of the vector fields: $\vec{F} = (z, 2yz, x + y^2)$.*

Solution: Let $f(x,y,z)$ be a potential function. Then

$$\vec{F} = (f_x, f_y, f_z) = (z, 2yz, x + y^2),$$

$$f_x = z, f_y = 2yz, f_z = x + y^2.$$

$$f_x = z \Rightarrow f = xz + g(y,z) \Rightarrow f_y = g_y = 2yz \Rightarrow g(y,z) = y^2z + h(z), f = xz + y^2z + h(z) \Rightarrow$$
$$f_z = x + y^2 + h'(z) = x + y^2, \Rightarrow h'(z) = 0 \Rightarrow h(z) = constant, C.$$

Hence,

$$f = xz + y^2z + C.$$

Example 62. *Let $f(x,y,z) = xyz \ln z$ be a potential function of $\vec{F}$. Find $\vec{F}$.*

Solution:

$$\vec{F} = (f_x, f_y, f_z) = (yz \ln z, xz \ln z, xy \ln z + xy).$$

Divergence and Curl

Divergence measures the magnitude of a vector field's source or sink at a given point, in terms of a signed scalar. The curl of a vector field measures how a fluid may rotate.

Let

$$\vec{F} = (P(x,y,z), Q(x,y,z), R(x,y,z)),$$
$$\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z} = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}),$$

the vector differential operator.

- The divergence of $\vec{F}$ is: div $\vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \nabla \cdot \vec{F}$.
- The curl of $\vec{F}$ is:

$$curl\vec{F} = \nabla\times\vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}.$$

- If curl $\vec{F} = \vec{0}$ at a point P, then $\vec{F}$ is said to be irrotational at P. The $\vec{F}$ is conservative.
- *div curl*$\vec{F} = 0$.

Example 63. *Let $\vec{F}(x,y,z) = (xz, xy^3, xyz) = xz\vec{i} + xy^3\vec{j} + xyz\vec{k}$. Find div $\vec{F}(x,y,z)$, div $\vec{F}(1,-2,-1)$, curl $\vec{F}(x,y,z)$, curl $\vec{F}(0,1,1)$.*

Solution: Let $P(x,y,z) = xz, Q(x,y,z) = xy^3, R(x,y,z) = xyz$. Then
div $\vec{F}(x,y,z) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = z + 3xy^2 + xy$,
div $\vec{F}(1,-2,-1) = -1 + 12 - 2 = 9$,
curl $\vec{F}(x,y,z) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & xy^3 & xyz \end{vmatrix} = (xz, x - yz, y^3)$,
curl $\vec{F}(0,1,1) = (0,-1,-1)$.

Chapter 4 Line Integrals

- Line integral $\int_C f(x,y)ds$: The area of a "fence" with C as the base, and the height is given by $f(x,y)$.

- The mass of the wire C, if f is the density of the wire.

4.1 Line Integrals with respect to arc length

1. Line Integrals of Scalar fields in 2-D

Let C be a smooth curve given by $x = x(t), y = y(t), a \leq t \leq b$, or equivalently, by the vector equation $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$. Let the density at a point (x,y) on C be $f(x,y)$. Subdivide this curve segment into a number of small segments. The weight of a small segment of the curve is approximately $f(x^*,y^*)\Delta s$, where (x^*,y^*) is a point in this segment, and Δs is the length of this small segment. The sum $\sum f(x^*,y^*)\Delta s$ is an approximation of the total weight of the curve segment. The total weight of C is $\lim_{\Delta s \to 0} \sum f(x^*,y^*)\Delta s$.

Definition 1. *If f is defined on a smooth curve C, then the line integral of f along C is*

$$\int_C f(x,y)ds = \lim_{\Delta s \to 0} \sum f(x^*,y^*)\Delta s$$

if the limit exists.

This is also called the line integral of type I.

Calculation of a line integral: If the smooth curve C is defined by parametric equations $x = x(t), y = y(t), a \leq t \leq b$, then

$$\int_C f(x,y)ds = \int_a^b f(x,y)\sqrt{[x'(t)]^2 + [y'(t)]^2}dt, \quad ds = \sqrt{[x'(t)]^2 + [y'(t)]^2}dt.$$

If C can be subdivide into a finite number of segments: $C = C_1 \bigcup C_2 \bigcup \cdots \bigcup C_n$, and smooth on each segment, then the line integral is calculated for each segment and the sum is the line integral of C:

$$\int_C f ds = \int_{C_1} f ds + \cdots + \int_{C_n} f ds.$$

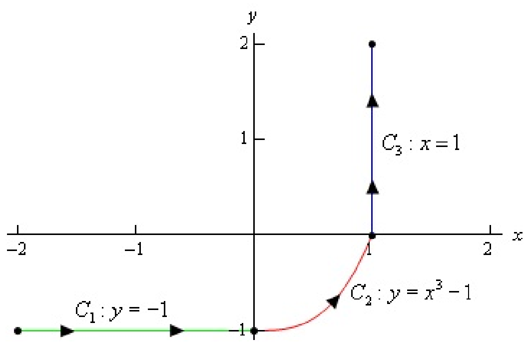
Example 64. *Find the mass of C with density $f(x,y) = y$, where C is the cycloid $x = t - \sin t, y = 1 - \cos t$, $0 \leq t \leq 2\pi$.*

Solution:

$$\int_C y ds = \int_0^{2\pi} y(t)\sqrt{[x'(t)]^2+[y'(t)]^2}dt = \int_0^{2\pi}(1-\cos t)\sqrt{2-2\cos t}dt$$

$$= \int_0^{2\pi} 4\sin^3(\frac{t}{2})dt = -8\left[\cos(t/2) - \frac{1}{3}\cos^3(t/2)\right]_0^{2\pi} = \frac{32}{3}.$$

Example 65. *Evaluate* $\int_C 4x^3 ds$*, where* C *is the curve shown below:*



Solution: The three curves are:

$C_1 : x = t, y = -1, -2 \le t \le 0$; $\quad C_2 : x = t, y = t^3 - 1, 0 \le t \le 1$; $\quad C_3 : x = 1, y = t, 0 \le t \le 2$.

$$\int_{C_1} 4x^3 ds = \int_{-2}^{0} 4t^3\sqrt{1^2+0^2}dt = -16,$$

$$\int_{C_2} 4x^3 ds = \int_0^1 4t^3\sqrt{1^2+(3t^2)^2}dt = \frac{2}{27}(10^{3/2}-1),$$

$$\int_{C_3} 4x^3 ds = \int_0^2 4(1)^3\sqrt{0^2+1^2}dt = 8,$$

thus

$$\int_C 4x^3 ds = \int_{C_1} 4x^3 ds + \int_{C_2} 4x^3 ds + \int_{C_3} 4x^3 ds = \frac{2}{27}(10^{3/2}-1) - 8.$$

Special cases: $ds = dx, or ds = dy$: $\int_C f(x,y)dx$ and $\int_C f(x,y)dy$ are called respectively line integral of f along C with respect to x and y. Suppose C is defined by parametric equations $x = u(t), y = v(t)$, $a \le t \le b$, then

$$\int_C P(x,y)dx + Q(x,y)dy = \int_a^b [P(u,v)u'(t) + Q(u,v)v'(t)]dt.$$

Example 66. *Find* $I = \int_C (x+y)dx + (x-y)dy$ *, where* C *is a curve defined by* $x = e^t \sin t, y = e^t \cos t, 0 \le t \le \pi/2$.

Solution:

$$I = \int_C (x+y)dx + (x-y)dy = \int_0^{\pi/2} [(e^t \sin t + e^t \cos t)(e^t \sin t)' + (e^t \sin t - e^t \cos t)(e^t \cos t)']dt$$

$$= \int_0^{\pi/2} 2e^{2t} \sin(2t)dt = \int_0^{\pi} e^w \sin(w)dw = \frac{1}{2}(e^\pi + 1).$$

2. Line Integrals of Scalar fields in 3-D

Calculation of a line integral: If the smooth curve C is defined by parametric equations $x = x(t), y = y(t), z = z(t), a \le t \le b$, then

$$\int_C f(x, y, z)ds = \int_a^b f(x, y, z)\sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}dt.$$

Example 67. *Find $\int_C (xy - z)ds$, where C is the cycloid $x = t, y = t^2, z = \frac{2}{3}t^3$, $0 \le t \le 1$.*

Solution:

$$\int_C (xy - z)ds = \int_0^1 (xy - z)\sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}dt = \int_0^1 \frac{1}{3}t^3\sqrt{1 + 4t^2 + 4t^4}dt = \frac{7}{36}.$$

4.2 Line Integrals of Vector fields

Definition 2. *Let $\vec{F}$ be a continuous vector field defined on a smooth curve $C : \vec{r} = \vec{r}(t), a \le t \le b$. Then the line integral of $\vec{F}$ along C is:*

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}) \cdot \vec{r}\,'(t)dt = \int_C \vec{F} \cdot \vec{T}ds,$$

which can be interpreted as the total work done by this force vector field when this object is moving from one end of C to the other end of C.

$$\int_C \vec{F} \cdot d\vec{r} = \int_C P(x, y)dx + Q(x, y)dy, \quad if \;\; \vec{F} = (P(x, y), Q(x, y));$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_C P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz,$$

if $\vec{F} = (P(x, y, z), Q(x, y, z), R(x, y, z))$.

Example 68. *Find* $\int_C \vec{F} \cdot d\vec{r}$, *where* $\vec{F} = (x, y, z)$ *and* C *is:* $x = \cos t, y = \sin t, z = \sin t$, $0 \leq t \leq \pi/2$.

Solution:

$$\int_C \vec{F} \cdot d\vec{r} = \int_C xdx + ydy + zdz = \int_0^{\pi/2} [(\cos t)(\cos t)' + (\sin t)(\sin t)' + (\sin t)(\sin t)']dt$$

$$= \int_0^{\pi/2} \sin t \cos t dt = \frac{1}{2}.$$

4.3 Line Integrals of Conservative Vector fields

Fundamental Theorem for Line Integral: Let $\vec{F} = \nabla f$ be a conservative vector field, and C be a smooth curve defined by parametric equation $C : \vec{r} = \vec{r}(t), a \leq t \leq b$. Then

$$\int_C \vec{F} \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)).$$

Path Independence of Line Integrals: Let $\vec{F}$ be a vector field continuous in an open connected region. The following statements are equivalent:

(a) The line integral is independent of path.

(b) The line integral is zero along any closed curve.

(c) $\vec{F}$ is a conservative vector field.

Example 69. *The vector field* $\vec{F} = (3+2xy, x^2-3y^2)$ *is conservative. A potential function of this field is* $f(x, y) = 3x + x^2y - y^3$. *Find* $\int_C (3 + 2xy)dx + (x^2 - 3y^2)dy$, *where* C *is a curve defined by* $x = e^t \sin t, y = e^t \cos t, 0 \leq t \leq \pi$.

Solution: To find the line integral, look at the starting and ending point of the curve. When $t = 0, x = 0, y = 1$. When $t = \pi, x = 0, y = -e^{\pi}$. Hence

$$\int_C (3+2xy)dx+(x^2-3y^2)dy = \int_C \vec{F} \cdot d\vec{r} = f(\vec{r}(\pi))-f(\vec{r}(0)) = f(0, -e^{\pi})-f(0, 1) = e^{3\pi}+1.$$

Example 70. *Given* $f(x, y, z) = xz + y^2z$ *is a potential function of the vector field* $\vec{F} = (z, 2yz, x+y^2)$. *Find* $\int_C \vec{F} \cdot d\vec{r}$, *where* C *is a curve defined by* $x = t, y = t^2, z = 2t, 0 \leq t \leq 1$.

Solution: To find the line integral, look at the starting and ending point of the curve. When $t = 0, x = 0, y = 0, z = 0$. When $t = 1, x = 1, y = 1, z = 2$. Hence

$$\int_C \vec{F} \cdot d\vec{r} = f(1,1,2) - f(0,0,0) = 4.$$

A new method to find f such that $\vec{F}(x,y,z) = \nabla f$:

$$f(x,y,z) = \int_0^1 \vec{F}(tx,ty,tz) \cdot (x,y,z)\ dt + c.$$

Proof. Let C be the line segment from $P_0(x_0,y_0,z_0)$ to $P(x,y,z)$, where P_0 is arbitrary.
$C : \vec{r}(t) = (1-t)P_0 + tP = ((1-t)x_0 + tx, 1-t)y_0 + ty, 1-t)z_0 + tz), 0 \le t \le 1.$
Usually we take $P_0 = (0,0,0)$. Then
$C : \vec{r}(t) = tP = (tx,ty,tz), 0 \le t \le 1.$

$$f(x,y,z) = \int_C \vec{F}(x,y,z) \cdot d\vec{r} + c = \int_0^1 \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\ dt + c$$
$$= \int_0^1 \vec{F}(tx,ty,tz) \cdot \vec{r}\,'(t)\ dt + c.$$

Example 71. *Find $f(x,y,z)$ such that $\nabla f = \vec{F} = (z, 2yz, x + y^2)$, and $f(1,1,-1) = 5$.*

Solution: Take $P_0 = (0,0,0)$ (P_0 is always arbitrary). Let $C : \vec{r}(t) = (1-t)(0,0,0) + t(x,y,z) = (tx,ty,tz), 0 \le t \le 1.$

$$f(x,y,z) = \int_C \vec{F} \cdot d\vec{r} + c$$

$$= \int_0^1 \vec{F}(tx,ty,tz) \cdot \vec{r}'(t)\ dt + c$$

$$= \int_0^1 \vec{F}(tx,ty,tz) \cdot (x,y,z)\ dt + c$$

$$= \int_0^1 (tz, 2tytz, tx + (ty)^2) \cdot (x,y,z)\ dt + c$$

$$= \int_0^1 (tzx + 2t^2y^2z + tzx + t^2y^2z)\ dt + c$$

$$= \int_0^1 (2tzx + 3t^2y^2z)\ dt + c$$

$$= xz + y^2z + c.$$

$$f(1,1,-1) = 5, \Rightarrow -2 + c = 5, \Rightarrow c = 7.$$

$$f(x,y,z) = xz + y^2z + 7.$$

Chapter 5: Double Integrals

5.1-5.2 Double Integrals

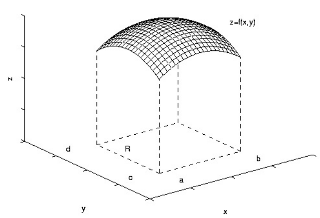
Consider the function $z = f(x, y)$ defined on a rectangle $R : a \leq x \leq b, c \leq y \leq d$. Subdivide $[a, b]$ into $a = x_0 < x_1 < \cdots < x_m = b$, and $[c, d]$ into $c = y_0 < y_1 < \cdots < y_n = d$.

The double integral over this rectangle is

$$\iint\limits_R f(x, y)dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}, y_{ij})\Delta A,$$

where $\Delta A = \Delta x \Delta y$, $\Delta x = \frac{b-a}{m}$, $\Delta y = \frac{d-c}{n}$, $x_{i-1} \leq x_{ij} \leq x_i$, $y_{i-1} \leq y_{ij} \leq y_i$.

Geometric meaning: If $f(x, y) \geq 0$, then it is the volume of the solid under the graph of $f(x, y)$, above the x-y plane, bounded by R.



The average value of the function defined inside R is

$$f_{ave} = \frac{1}{(b-a)(d-c)} \iint\limits_R f(x, y)dA.$$

Numerical Approximation: Midpoint Rule: To approximate a double integral numerically, we may choose the middle point in each small rectangle as the sample point, or the top right corner (x_i, y_j) as the sample point.

Properties of Double Integrals:

(i) $\iint\limits_R [f(x, y) + g(x, y)]dA = \iint\limits_R f(x, y)dA + \iint\limits_R g(x, y)dA$.

(ii) $\iint\limits_R cf(x, y)dA = c \iint\limits_R f(x, y)dA$.

(iii) If $f(x, y) \leq g(x, y)$ for all $(x, y) \in R$, then $\iint\limits_R f(x, y)dA \leq \iint\limits_R g(x, y)dA$.

(iv) If $R = R_1 \bigcup R_2$, $R_1 \bigcap R_2 = \emptyset$, then $\iint\limits_R f(x, y)dA = \iint\limits_{R_1} f(x, y)dA + \iint\limits_{R_2} f(x, y)dA$.

(v) $\iint\limits_R dA =$ the area of R.

(vi) If $m \leq f(x, y) \leq M$ for all $(x, y) \in R$, then $mA \leq \iint\limits_R f(x, y)dA \leq MA$, where A is the area of R.

Iterated Integrals

Fubini's Theorem: If $f(x,y)$ is continuous on the rectangle $R: a \le x \le b, c \le y \le d$, then

$$\iint\limits_R f(x,y)dA = \int_a^b \int_c^d f(x,y)dy\,dx = \int_c^d \int_a^b f(x,y)dx\,dy.$$

The right hand side is called iterated integral. By this theorem, we can evaluate a double integral using an iterated integral.

Special case: If $f(x,y) = g(x)h(y)$, then the iterated integral becomes the product of two integrals.

$$\iint\limits_R f(x,y)dA = \left(\int_a^b g(x)\,dx\right)\left(\int_c^d h(y)\,dy\right).$$

Example 72. *Find* $\iint\limits_R zdA$*, where* $z = y\sin(xy)$*,* $R = \{(x,y)|1 \le x \le 2, 0 \le y \le \pi/2\}$.

Solution:

$$\iint\limits_R f(x,y)dA = \int_0^{\pi/2}\int_1^2 y\sin(xy)dx\,dy = \int_0^{\pi/2} \left(-\cos(xy)\right)|_1^2 dy$$

$$= \int_0^{\pi/2} \left(-\cos(2y) + \cos(y)\right)dy = \left(-\frac{1}{2}\sin(2y) + \sin(y)\right)|_0^{\pi/2} = 1.$$

Remark. We may use the other order to integrate with respect to x first, but it involves an integral that is harder to evaluate.

Example 73. *Find* $\iint\limits_R zdA$*, where* $z = 16 - x^2 - 2y^2$*,* $R = [0,2] \times [0,2]$.

Solution:

$$\iint\limits_R zdA = \int_0^2\int_0^2 (16 - x^2 - 2y^2)dx\,dy = \int_0^2 \left(16x - \frac{1}{3}x^3 - 2xy^2\right)|_{x=0}^2 dy$$

$$= \int_0^2 \left(\frac{88}{3} - 4y^2\right)dy = 48.$$

Double Integrals over General Regions

y-simple (or Type I): A region R is of Type I, if

$$R = R_{yx} = \{(x,y) : a \le x \le b, g_1(x) \le y \le g_2(x)\}.$$

Then

$$\iint\limits_R f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dy\,dx.$$

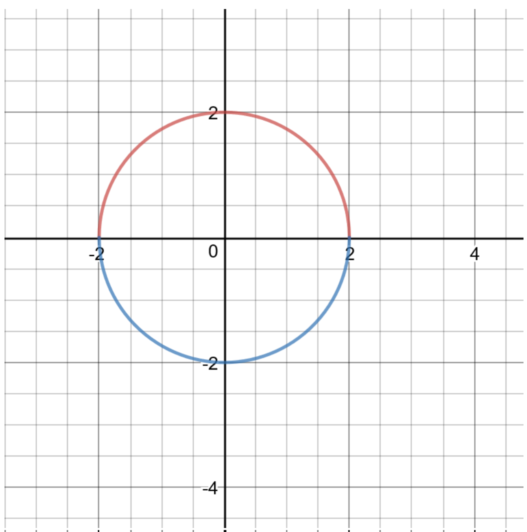
x-simple (or Type II): A region R is of Type II, if

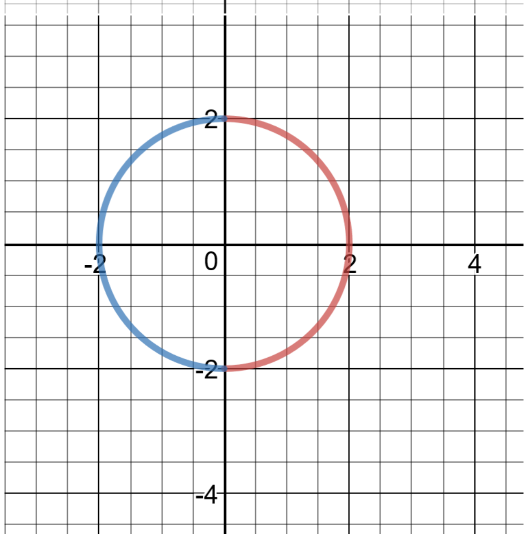
$$R = R_{xy} = \{(x,y) : c \le y \le d, h_1(y) \le x \le h_2(y)\}.$$

Then

$$\iint\limits_R f(x,y)dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dx\,dy.$$

Example 74. *A region $D:\ x^2+y^2 \le 4$. Rewrite it as y-simple, and x-simple region.*



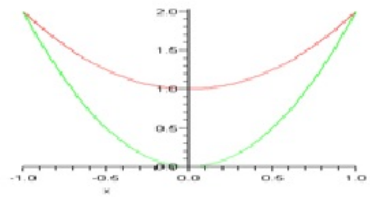


Solution:

- y-simple: $-\sqrt{4-x^2} \le y \le \sqrt{4-x^2}$, $-2 \le x \le 2$.

- x-simple: $-\sqrt{4-y^2} \le x \le \sqrt{4-y^2}$, $-2 \le y \le 2$.

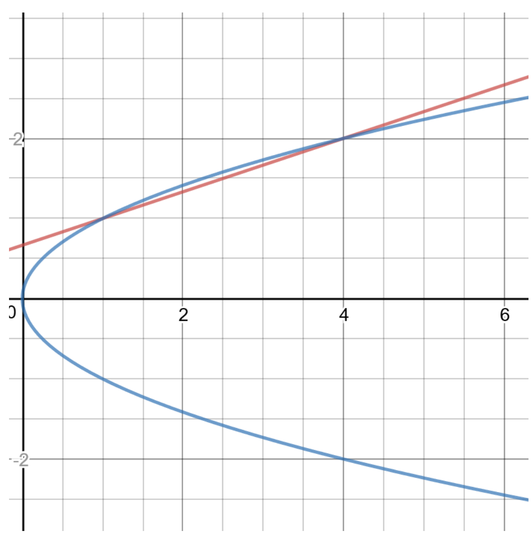
Example 75. *Find* $\iint\limits_R z dA$*, where* $z = x + 2y$*,* R *is the region bounded by* $y = 2x^2$ *and* $y = x^2 + 1$.



Solution: The intersection points of $y = 2x^2$ and $y = x^2 + 1$ are $x = \pm 1$. Thus $R = \{(x, y) : -1 \le x \le 1, 2x^2 \le y \le x^2 + 1\}$.

$$\iint\limits_R z dA = \int_{-1}^{1} \int_{2x^2}^{x^2+1} (x + 2y) dy\, dx = \int_{-1}^{1} (-3x^4 - x^3 + 2x^2 + 1)\, dx = \frac{32}{15}.$$

Example 76. *Evaluate* $\iint_D \ln y \; dA$*,* D *is the region bounded by* $3y = x + 2$ *and* $x = y^2$.



Solution: The intersections are $(1, 1), (4, 2)$. Thus $D = R_{xy} = \{(x, y) : 1 \le y \le 2, y^2 \le x \le 3y - 2\}$. Thus

$$\iint_D \ln y \; dA = \int_1^2 \int_{y^2}^{3y-2} \ln y \; dx \; dy$$

$$= \int_1^2 (3y - 2 - y^2) \ln y \; dy$$

$$= \left[(\frac{3}{2}y^2 - 2y - \frac{1}{3}y^3) \ln y - (\frac{3}{4}y^2 - 2y - \frac{1}{9}y^3) \right]_1^2$$

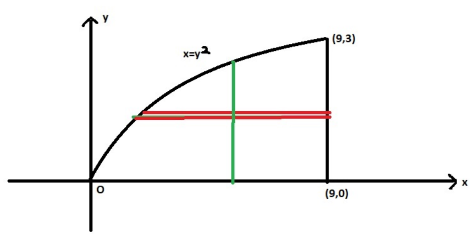
$$= -\frac{2}{3}\ln 2 + \frac{19}{36}.$$

General Region:

A region R has to be subdivided into a number of such regions and calculate the integral separately. If a region can be regarded either as a Type I region, or a Type II region, in some cases, the order of integration is significant.

Changing the order of integration: Some regions can be regarded as of Type I or of Type II. We may use two different ways to express a double integral over such a region as iterated integral. In some cases, both ways are appropriate, and give the same result. However, in some cases, one iterated integral can be evaluated, but the other cannot.

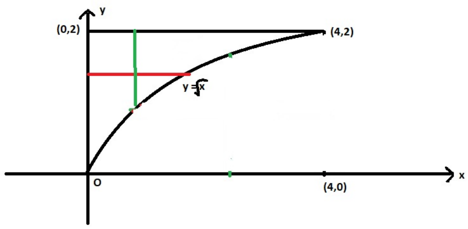
Example 77. *Sketch and shade the region of the integral* $\int_0^3 \int_{y^2}^9 y\sin(x^2)dxdy$*, then evaluate the integral.*



Solution: Since the integral $\int_{y^2}^9 y\sin(x^2)dx$ cannot be integrated analytically, this iterated integral cannot be integrated in this order. We need to change the order.

$$\int_0^3 \int_{y^2}^9 y\sin(x^2)dxdy = \iint_R y\sin(x^2)dA = \int_0^9 \int_0^{\sqrt{x}} y\sin(x^2)dydx = \frac{1-\cos 81}{4}.$$

Example 78. *Sketch and shade the region of the integral* $\int_0^4 \int_{\sqrt{x}}^2 \sqrt{1+y^3}\ dydx$*, then evaluate the integral.*



Solution:

$$\{(x,y) : \sqrt{x} \leq y \leq 2, 0 \leq x \leq 4\} \rightarrow \{(x,y) : 0 \leq y \leq 2, 0 \leq x \leq y^2\}$$

$$\int_0^4 \int_{\sqrt{x}}^2 \sqrt{1+y^3}\ dydx = \int_0^2 \int_0^{y^2} \sqrt{1+y^3}\ dx\ dy$$

$$= \int_0^2\ y^2\sqrt{1+y^3}\ dy$$

$$\int_1^9 \frac{1}{3}\ \sqrt{u}\ du, \quad u = 1+y^3$$

$$= \frac{2}{9}u^{3/2}|_1^9 = \frac{52}{9}.$$

5.3 Applications of Double Integrals

1. Volume of solids under a surface: Let $z = f(x,y)$, $(x,y) \in R$ define a surface S, where $f(x,y) \geq 0$ for all $(x,y) \in D$. Then the volume V of the solid lying directly above D is:

$$V = \iint_D f(x,y)dA.$$

Example 79. *Find the volume under the surface $z = 4 - x^2 - y^2$ that projects onto the region*

$$R = \{(x,y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}.$$

Solution:

$$V = \iint_R zdA = \int_0^1 \int_0^1 (4 - x^2 - y^2)dx\,dy = \int_0^1 (4 - x^2 - \frac{1}{3})dx = 4 - \frac{2}{3} = \frac{10}{3}.$$

Volume of solids of revolution: The volume of the solid of revolution obtained by rotating D about the x-axis in the plane is given by

$$\iint_D 2\pi y dA.$$

The volume of the solid of revolution obtained by rotating D about the y-axis in the plane is given by

$$\iint_D 2\pi x dA.$$

Example 80. *Find the volume of the sphere with radius r.*

Solution: $\frac{4}{3}\pi r^3$.

2. Mass of the lamina: Let $z = \rho(x, y)$, $(x, y) \in D$ define the density of the lamina occupies the region D. Then the mass m of the lamina is:

$$m = \iint_D \rho(x, y) dA.$$

Example 81. *Find the mass of the lamina that occupies the region $D = \{(x, y) : 0 \le x \le 1, 0 \le y \le 1\}$ with the density function $\rho(x, y) = xye^{x^2}$.*

Solution:

$$m = \iint_D \rho(x, y) dA = \int_0^1 \int_0^1 xye^{x^2}\ dx\ dy$$

$$\int_0^1 \left[\frac{1}{2} ye^{x^2}\right]_{x=0}^1 dy = \int_0^1 \frac{e-1}{2} y\ dy = \frac{e-1}{4}.$$

Moments and Centers of Mass: Now lets find the center of mass of a lamina with density function $\rho(x, y)$ that occupies a region D.

Recall that the moment of a particle about an axis is defined as the product of its mass and its directed distance from the axis. The moments of the entire lamina about the x-axis and about the y-axis are:

$$M_x = \iint_D y\rho(x, y) dA, \quad M_y = \iint_D x\rho(x, y) dA.$$

The center of mass of a lamina occupying the region D and having density function $\rho(x, y)$ and mass m are:

$$(\bar{x}, \bar{y}) = (\frac{M_y}{m}, \frac{M_x}{m}).$$

Example 82. *Let* $\rho(x, y) = x^2+y^2$*, and* D *is a triangle bounded by* $x = 0$*,* $x = y$*,* $x+y = 2$*. Find the mass of the lamina and the center of mass.*

Solution: $m = \frac{4}{3}$, $M_y = \frac{7}{15}$, $M_x = \frac{5}{3}$.

3. Surface area: For a differentiable surface S: $z = f(x, y)$, $(x, y) \in D$, the area of the surface is

$$\iint_S dS = \iint_D \sqrt{(f_x)^2 + (f_y)^2 + 1}\ dA.$$

$$dS = \sqrt{(f_x)^2 + (f_y)^2 + 1}\ dA.$$

Example 83. *Find the area of the surface* $z = 2y + \frac{2}{3}x^{3/2}$ *that lies directly above the region* $D = \{(x, y) : 4 \le x \le 11, 0 \le y \le 3\}$*.*

Solution:

$$Surface\ Area\ = \iint_D \sqrt{(z_x)^2 + (z_y)^2 + 1}\ dA$$

$$= \iint_D \sqrt{(\sqrt{x})^2 + 2^2 + 1}\ dA = \iint_D \sqrt{x+5}\ dA$$

$$\int_4^{11} \int_0^3 \sqrt{x+5}dy\ dx = \int_4^{11} 3\sqrt{x+5}dx$$

$$= 2(x+5)^{3/2}\Big|_4^{11} = 2(64-27) = 74.$$

5.4 Change of variables in double integrals

Review Substitution rule in calculus I.

Change of Variables for a Double Integral: Suppose that we want to integrate $f(x, y)$ over the region R. Under the change of variables: $x = g(u, v), y = h(u, v)$, the region R for (x, y) becomes D for (u, v), and the integral becomes,

$$\iint_R f(x, y)dA = \iint_D f(g, h)J(u, v)dudv,$$

where

- $J(u, v) = \left|\frac{\partial(x, y)}{\partial(u, v)}\right| = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}$, and is called the Jacobian of the transformation.

- $dA = J(u,v)dudv$.
- For polar coordinates $x = x_0 + r\cos\theta,\ y = y_0 + r\sin\theta,\ J(r,\theta) = r$, where (x_0, y_0) is the centre.

$$\iint\limits_R f(x,y)dA = \iint_D f(x_0 + r\cos\theta, y_0 + r\sin\theta)\ r\ dr\ d\theta.$$

- For elliptic coordinates $x = x_0 + ar\cos\theta,\ y = y_0 + br\sin\theta,\ J(r,\theta) = abr$, where (x_0, y_0) is the centre.

$$\iint\limits_R f(x,y)dA = \iint_D f(x_0 + ar\cos\theta, y_0 + br\sin\theta)\ abr\ dr\ d\theta.$$

Example 84. *Evaluate* $\iint\limits_R zdA$*, where* $z = e^{-((x+2)^2+y^2)}$*,*
$R = \{(x,y) : x \geq -2, y \geq 0, x^2 + 4x + y^2 \leq 0\}$.

Solution: In polar coordinates: $x = -2 + r\cos\theta$, $y = r\sin\theta$,
$R \Rightarrow D = \{(r,\theta) : 0 \leq \theta \leq \pi/2, 0 \leq r \leq 2\}$

$$\iint\limits_R zdA = \int_0^{\pi/2}\int_0^2 e^{-(r^2)} r dr d\theta = \frac{(1-e^{-4})\pi}{4}.$$

Example 85. *Find the volume of the sphere* $x^2 + y^2 + z^2 = a^2$*.*

Solution: Let $R = \{(x,y) : x^2 + y^2 \leq a^2\}$. Then

$$V = 2\iint_R z\ dA = 2\iint_R \sqrt{a^2 - x^2 - y^2}\ dA$$

$$= 2\iint_D \sqrt{a^2 - r^2} r d\theta dr = 2\int_0^a \int_0^{2\pi} r\sqrt{a^2 - r^2} d\theta dr$$

$$= 4\pi \int_0^a r\sqrt{a^2 - r^2} dr = -2\pi \int_{a^2}^0 \sqrt{u} du, \quad u = a^2 - r^2$$

$$= -2\pi \frac{2}{3} u^{3/2}|_{a^2}^0 = \frac{4}{3}\pi a^3.$$

Example 86. *Find the surface area of the sphere* $x^2 + y^2 + z^2 = a^2$*.*

Solution: Let $R = \{(x, y) : x^2 + y^2 \le a^2\}$, $z = \sqrt{a^2 - x^2 - y^2}$. Then

$$z_x = -\frac{x}{\sqrt{a^2 - x^2 - y^2}}, \quad z_y = -\frac{y}{\sqrt{a^2 - x^2 - y^2}}.$$

$$\begin{aligned} surface\ area &= 2\iint_R \sqrt{z_x^2 + z_y^2 + 1}\, dA = 2\iint_R \frac{a}{\sqrt{a^2 - x^2 - y^2}}\, dA \\ &= 2\iint_D \frac{a}{\sqrt{a^2 - r^2}} r d\theta dr, \quad x = r\cos\theta, y = r\sin\theta \\ &= 2a\int_0^a \int_0^{2\pi} \frac{r}{\sqrt{a^2 - r^2}} d\theta dr = 4\pi a \int_0^a \frac{r}{\sqrt{a^2 - r^2}} dr \\ &= -2\pi a \int_{a^2}^0 u^{-1/2} du, \quad u = a^2 - r^2 \\ &= 4\pi a^2. \end{aligned}$$

Example 87. *Find* $\iint\limits_D (x-2)(y+1)dA$, *where*
$D = \{(x, y) : 36x^2 - 144x + y^2 + 2y + 109 \le 0, x \ge 2, y \ge -1\}$.

Solution: Rewrite the region D:

$$36(x-2)^2 + (y+1)^2 \le 36.$$

Let $x = 2 + r\cos\theta, y = -1 + 6r\sin\theta$. $0 \le r \le 1$, $0 \le \theta \le \pi/2$.

$$\begin{aligned} \iint_R xydA &= \int_0^{\pi/2}\int_0^1 6r^2\cos\theta\sin\theta\ 6rdrd\theta \\ &= 36\left(\frac{1}{4}r^4\right)\Big|_0^1 \left(\frac{1}{2}\sin^2\theta\right)\Big|_0^{\pi/2} = \frac{9}{2}. \end{aligned}$$

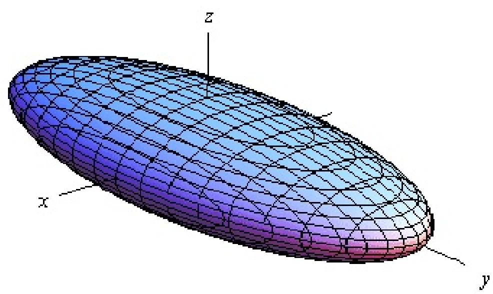
5.5 3-D plots

A **quadric surface** is the graph of a second degree equation in three variables x, y, z:

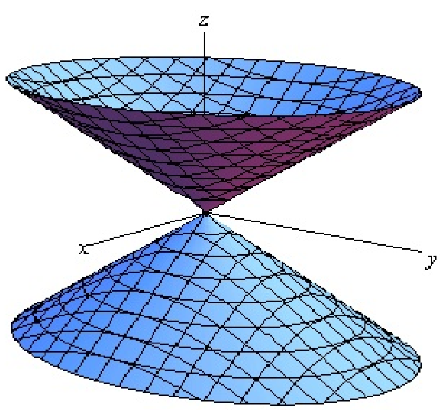
$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0.$$

By rotating the surface, or equivalently, rotating the axes, we may assume that $D = E = F = 0$.

1. Ellipsoid: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, $a, b, c > 0$.



2. (Elliptic) Cone: The general equation of a cone is: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$, $a, b, c > 0$.



3. Cylinder

A cylinder is a surface consisting of lines parallel (rulings) to a given line passing through a given plane curve. In most cases, the rulings are parallel to an axis. In this case, the equation of the surface contains only two variables, which gives the plane curve in a coordinate plane. For example,

(i) (Right-circular) Cylinder $x^2 + y^2 = a^2$. This is a cylinder containing lines parallel to the z-axis passing through points on the unit circle in the xy- plane.

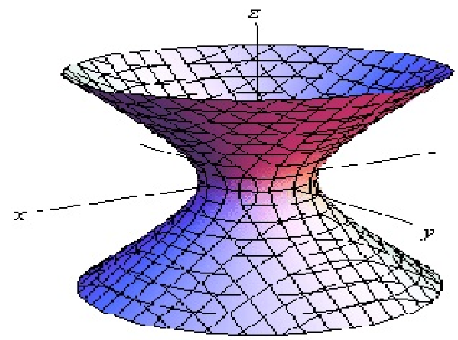
(ii) Parabolic cylinder $x^2 + 2rz = 0$, $r \neq 0$. This is a cylinder containing lines parallel to the y-axis passing through points on the curve $z = x^2$ in the xz- plane.

(iii) $yz = 1$. This is a cylinder containing lines parallel to the x-axis passing through points on two branches of the curve $yz = 1$ in the yz- plane.

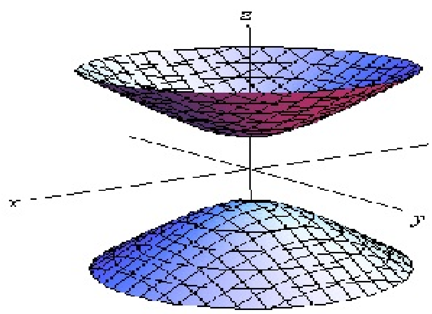
(iv) Elliptic cylinder $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. This is a cylinder containing lines parallel to the z-axis passing through points on the ellipse in the xy- plane.

(v) Hyperbolic cylinder $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. This is a cylinder containing lines parallel to the z-axis passing through points on the ellipse in the xy- plane.

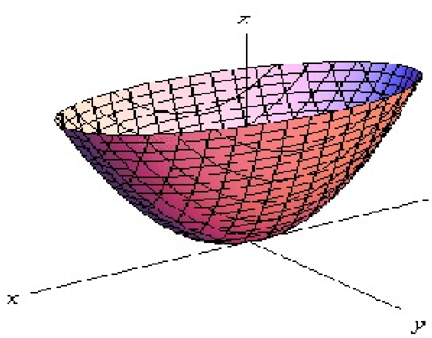
4. Hyperboloid of One Sheet: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$, $a, b, c > 0$.



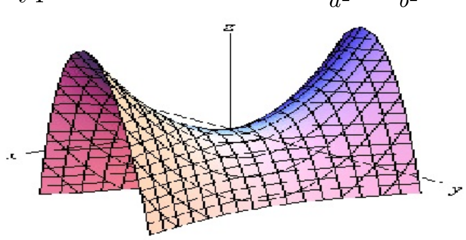
5. Hyperboloid of Two Sheets: $-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, $a, b, c > 0$.



6. Elliptic Paraboloid: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$, $a, b, c > 0$.



7. Hyperbolic Paraboloid: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{z}{c}$, $a, b, c > 0$.



5.6 Parametric equations of surfaces

Let $\vec{r}(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}$, $(u,v) \in D$. Then $\{(x,y,z) : x = x(u,v), y = y(u,v), z = z(u,v), (u,v) \in D\}$ is called a parametric surface S represented by $\vec{r}$.

Some special surfaces:

- Ellipsoid
  1. Cartesian (rectangular) equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$.
  2. Parametric form $x = a\sin\phi\cos\theta, y = b\sin\phi\sin\theta, z = c\cos\phi$, where $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi$.
- Elliptic cone
  1. Cartesian equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 0$.
  2. Parametric form $x = av\cos\theta, y = bv\sin\theta, z = cv$, where $0 \le \theta \le 2\pi$, $v \in \mathbb{R}$.
- Elliptic cylinder
  1. Cartesian equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.
  2. Parametric form $x = a\cos\theta, y = b\sin\theta, z = v$, where $0 \le \theta \le 2\pi$.
- Hyperbolic cylinder
  1. Cartesian equation $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = -1$.
  2. Parametric form $x = a\sinh u, y = b\cosh u, z = v$, where $u, v \in \mathbb{R}$.
- Hyperbolic paraboloid
  1. Cartesian equation $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = -z$.
  2. Parametric form $x = a\sqrt{v}\sinh u, y = b\sqrt{v}\cosh u, z = v$, where $u, v \in \mathbb{R}$.
- Torus
  1. Cartesian equation $x^2 + y^2 + z^2 + c^2 - a^2 - 2c\sqrt{x^2 + y^2} = 0$.
  2. Parametric form $x = (c + a\cos v)\cos u$, $y = (c + a\cos v)\sin u$, $z = a\sin v$, where $0 \le u, v < 2\pi$, $c > a > 0$.

Two special coordinates:

- Cylindrical coordinates: $x = r\cos\theta, y = r\sin\theta, z = z$.
- Spherical coordinates: $x = \rho\sin\phi\cos\theta, y = \rho\sin\phi\sin\theta, z = \rho\cos\phi$.

Example 88. *Ellipsoid*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

can be parameterized as

$$x = a\sin\phi\cos\theta, y = b\sin\phi\sin\theta, z = c\cos\phi,$$

where $0 \le \theta \le 2\pi$*,* $0 \le \phi \le \pi$*.*

Tangent plane. The tangent plane of the parametric surface S at a point (u_0, v_0) is the plane containing the two tangent vectors $\vec{r}_u(u_0, v_0)$ and $\vec{r}_v(u_0, v_0)$.

Surface area. Let S be the smooth surface above. If S is covered just once as (u, v) varies throughout D, then the surface area of S is

$$A(S) = \iint_D |\vec{r}_u \times \vec{r}_v| dA,$$

where

$$\vec{r}_u = \frac{\partial x}{\partial u}\vec{i} + \frac{\partial y}{\partial u}\vec{j} + \frac{\partial z}{\partial u}\vec{k}, \quad \vec{r}_v = \frac{\partial x}{\partial v}\vec{i} + \frac{\partial y}{\partial v}\vec{j} + \frac{\partial z}{\partial v}\vec{k}.$$

Example 89. *Find the surface area of a sphere with radius* r*.*

Solution: The parametric surface is: $x = r\sin\phi\cos\theta, y = r\sin\phi\sin\theta, z = r\cos\phi$, where $(\phi, \theta) \in D = \{0 \le \phi \le \pi, 0 \le \theta \le 2\pi\}$.

$$\vec{r}_\phi \times \vec{r}_\theta = r^2(\sin^2\phi\cos\theta, \sin^2\phi\sin\theta, \sin\phi\cos\phi),$$

$$|\vec{r}_\phi \times \vec{r}_\theta| = r^2\sin\phi.$$

$$A(S) = \iint_D |\vec{r}_\phi \times \vec{r}_\theta| dA = 4\pi r^2.$$

Example 90. *Find the area of the surface* $z = x^2 + y^2$ *that lies under the plane* $z = 4$*.*

Solution: Projecting the surface to xy-plane, we get $D: \ x^2 + y^2 \leq 4$.

$$A = \iint_D \sqrt{1 + z_x^2 + z_y^2} dA = \iint_D \sqrt{1 + 4(x^2 + y^2)} dA$$

$$= \int_0^{2\pi} \int_0^2 \sqrt{1 + 4r^2} \ r dr d\theta$$

$$= 2\pi \ \frac{1}{12} \left(1 + 4r^2\right)^{3/2} \Big|_0^1 = \frac{\pi}{6}(5\sqrt{5} - 1).$$

Chapter 6: The Three Big Theorems

6.1-6.2 Surface Integrals

1. Surface Integrals of Scalar Fields

Let $f(x, y, z)$ be a function defined in a region in space containing a surface S. The surface integral of f over S is

$$\iint_S f(x, y, z)dS.$$

- If S is defined by $z = g(x, y)$ and the projection of S onto the xy-plane is D, then

$$\iint_S f(x, y, z)dS = \iint_D f(x, y, g(x, y))\sqrt{g_x^2 + g_y^2 + 1}dA.$$

- If S is defined by $y = g(x, z)$ and the projection of S onto the xz-plane is D, then

$$\iint_S f(x, y, z)dS = \iint_D f(x, g(x, z), z)\sqrt{g_x^2 + g_z^2 + 1}dA.$$

- If S is defined by $x = g(y, z)$ and the projection of S onto the yz-plane is D, then

$$\iint_S f(x, y, z)dS = \iint_D f(g(y, z), y, z)\sqrt{g_y^2 + g_z^2 + 1}dA.$$

- Parametric Surfaces: Let S be be a smooth surface with parametric representation $\vec{r}(u, v) = x(u, v)\vec{i} + y(u, v)\vec{j} + z(u, v)\vec{k}$, $(u, v) \in D$. Then

$$\iint_S f(x, y, z)dS = \iint_D f(\vec{r}(u, v))|\vec{r}_u \times \vec{r}_v|dA.$$

The unit normal vector of the surface S is

$$\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}.$$

Example 91. *Evaluate $\iint_S ydS$, where S is the part of the plane $2x + 2y + z = 8$ that lies in the first octant .*

Solution: R is the triangle bounded by lines $x = 0$, $y = 0$, $2x + 2y = 8$. Note that

$$R = R_{yx} = \{(x, y) : 0 \le y \le 4 - x, 0 \le x \le 4\}.$$

By $z = 8 - 2x - 2y$, $z_x = -2$, $z_y = -2$. Thus

$$\iint_S ydS = \iint_R y\sqrt{z_x^2 + z_y^2 + 1}dA = \iint_R y\sqrt{(-2)^2 + (-2)^2 + 1}dA$$

$$= 3\int_0^4\int_0^{4-x} ydydx = \frac{3}{2}\int_0^4 (4-x)^2dx = 32.$$

Example 92. *Evaluate* $\iint_S \frac{x+y}{\sqrt{2z+1}}dS$*, where* S *is the surface given by* $\vec{r}(u,v) = (u+v)\vec{i} + (u-v)\vec{j} + (u^2+v^2)\vec{k}$, $x = u+v$, $y = u-v$, $z = u^2+v^2$, $(u,v) \in D = \{0 \le u \le 1, 0 \le v \le 2\}$.

Solution: $\vec{r}_u \times \vec{r}_v = 2(u+v)\vec{i} + 2(u-v)\vec{j} - 2\vec{k}$. Thus

$$dS = |\vec{r}_u \times \vec{r}_v| dA = 2\sqrt{2(u^2+v^2)+1}\ dA.$$

Thus

$$\iint_S \frac{x+y}{\sqrt{2z+1}}dS = \iint_D \frac{2u}{\sqrt{2(u^2+v^2)+1}}\ 2\sqrt{2(u^2+v^2)+1}\ dA$$

$$= \iint_D 4u\ dA = 4\int_0^2\int_0^1 u\, dudv = 4.$$

2. Surface Integrals of Vector Fields:

A surface S is orientable or two-sided if it has a unit normal vector $\vec{n}$ that varies continuously over S. Let $\vec{F}$ be a continuous vector field defined in a region containing an oriented surface S with unit normal vector $\vec{n}$. The surface integral (also called **flux integral**) of $\vec{F}$ across S in the direction of $\vec{n}$ is

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n} dS.$$

Remark: **flux** is the amount of "something" crossing a surface, such as water, wind, electric field.

If $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$, and S is given by $z = f(x,y)$ oriented upward and D is the projection onto the xy-plane, then

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D (-Pz_x - Qz_y + R)dA.$$

Example 93. *Find the flux of the vector field $\vec{F} = x\vec{i} + y\vec{j} + z\vec{k}$, S is the surface that is composed of the part of the paraboloid $z = \sqrt{4 - x^2 - y^2}$ lying inside $x^2 + y^2 = 1$.*

Solution: Project S to xy-plane, we get $D:\ x^2 + y^2 \leq 1$. Change D to $R = \{(r, \theta) : 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi\}$.

$$flux = \iint_S \vec{F} \cdot d\vec{S} = \iint_D (-Pz_x - Qz_y + R)dA = \iint_D \frac{1}{\sqrt{1 - x^2 - y^2}} dA$$

$$= \iint_R \frac{1}{\sqrt{1 - r^2}} r dr d\theta.$$

Parametric Surfaces: Let S be be a smooth oriented surface with parametric representation $\vec{r}(u, v) = x(u, v)\vec{i} + y(u, v)\vec{j} + z(u, v)\vec{k}$, $(u, v) \in D$.

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n} dS = \iint_D \vec{F}(\vec{r}(u, v)) \cdot (\vec{r}_u \times \vec{r}_v) dA.$$

Example 94. *Find the flux of the vector field $\vec{F} = z\vec{i} + y\vec{j} + x\vec{k}$ across the unit sphere $x^2 + y^2 + z^2 = 1$.*

Solution: The parametric surface is: S: $\vec{r}(\phi, \theta) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$, where $(\phi, \theta) \in D = \{0 \leq \phi \leq \pi, 0 \leq \theta \leq 2\pi\}$.

$$\vec{r}_\phi \times \vec{r}_\theta = (\sin^2\phi\cos\theta, \sin^2\phi\sin\theta, \sin\phi\cos\phi),$$

$$\vec{F} \cdot (\vec{r}_\phi \times \vec{r}_\theta) = 2\sin^2\phi\cos\phi\cos\theta + \sin^3\phi\sin^2\theta.$$

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F}(\vec{r}(u, v)) \cdot (\vec{r}_u \times \vec{r}_v) dA = \frac{4\pi}{3}.$$

6.3 Green's Theorem

Green's theorem gives the relationship between a line integral around a simple closed curve C and a double integral over the plane region D bounded by C.

Green's Theorem. Let C be a positively oriented (counter-clockwise), piecewise smooth, simple closed curve in the plane, and let D be the region bounded by C. Let $\vec{F}(x, y) =$

$(P(x,y),Q(x,y))$. If P and Q are functions of (x,y) defined on an open region containing D and have continuous partial derivatives there, then

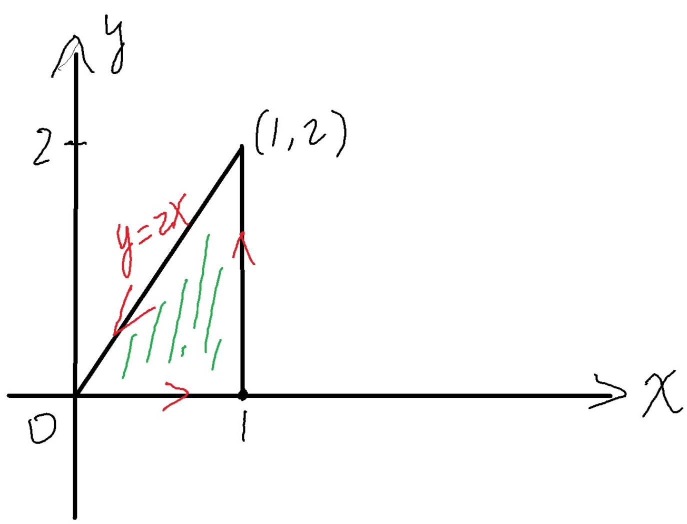
$$\oint_C Pdx + Qdy = \oint_C \vec{F}\cdot d\vec{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy.$$

Properties:

- If $\vec{F}$ is conservative, then $\oint_C \vec{F}\cdot d\vec{r} = 0$.
- For a closed region R with boundary C, the area of R is:

$$A(R) = \frac{1}{2}\int_C xdy - ydx.$$

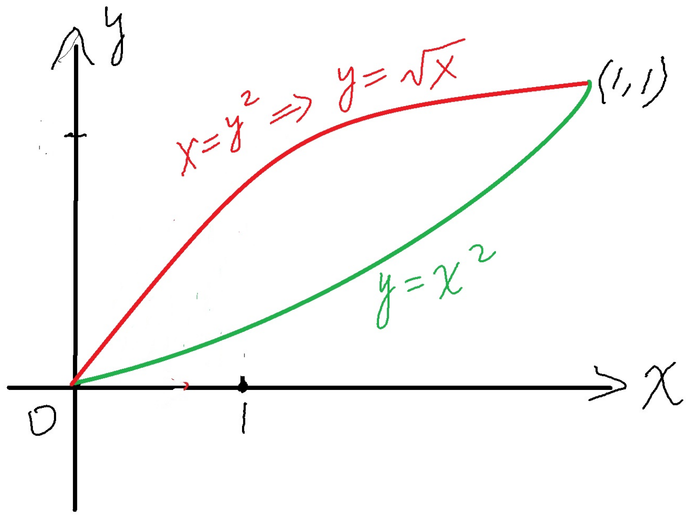
Example 95. *Use Green's Theorem to evaluate* $\oint_C xydx + x^2y^3dy$ *where* C *is the triangle with vertices* $(0,0),(1,0),(1,2)$*, with positive orientation.*



Solution: D is the triangle with vertices $(0,0),(1,0),(1,2)$. Then $D = R_{yx} : 0 \le y \le 2x, 0 \le x \le 1$.

$$\oint_C xydx + x^2y^3dy = \iint_D (2xy^3 - x)dxdy = \int_0^1 (\int_0^{2x} (2xy^3 - x)dy)dx = \frac{2}{3}.$$

Example 96. *Use Green's Theorem to evaluate* $\int_C (\sin x + y^3)dx + e^{y^2}dy$*, where* C *is the perimeter of the bounded region bounded by* $x = y^2$ *and* $y = x^2$ *with positive orientation.*



Solution: Note that $Q_x - P_y = -3y^2$. By Green's Theorem,

$$\int_C (\sin x + y^3)dx + e^{y^2}dy = \iint_D (-3y^2)dA.$$

To find D, the intersection between $x = y^2$ and $y = x^2$: (0,0), (1,1). Thus

$$D: \; x^2 \le y \le \sqrt{x}, \; 0 \le x \le 1.$$

Hence

$$\int_C (\sin x + y^3)dx + e^{y^2}dy = \iint_D (-3y^2)dA = \int_0^1 \int_{x^2}^{\sqrt{x}} (-3y^2)dydx$$

$$= \int_0^1 (-y^3)|_{y=x^2}^{\sqrt{x}} dx = \int_0^1 (-x^{3/2} + x^6)dx = (-\frac{2}{5}x^{5/2} + \frac{1}{7}x^7)|_0^1 = -\frac{9}{35}.$$

6.4 Stokes' Theorem

Stokes' theorem relates a line integral over a closed curve to a surface integral. It is a generalization of Green's Theorem to higher dimension.

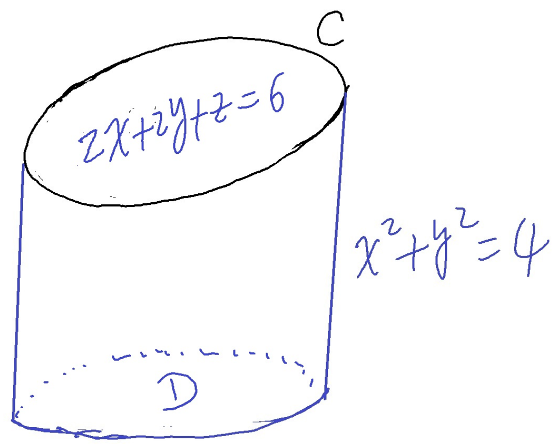
Let S be an oriented piecewise-smooth surface that has a unit normal vector $\vec{n}$ and is bounded by a simple closed positively oriented curve C. If $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ is a vector field, where P, Q, R have continuous partial derivatives on an open region containing S, then

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot d\vec{S}.$$

Physical interpretation: If $\vec{F}$ is a force field, then the work done by $\vec{F}$ along C = the flux of curl$\vec{F}$ across S.

Remark $\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \iint_S curl\vec{F} \cdot d\vec{S} = \iint_S curl\vec{F} \cdot \vec{n} dS.$

Example 97. *Let $\vec{F} = z\vec{i} + x^2\vec{j} + 2y\vec{k}$, and let S be the surface whose boundary C is the curve of intersection of the plane $2x + 2y + z = 6$ and the cylinder $x^2 + y^2 = 4$. Using Stokes' Thm evaluate $\int_C \vec{F} \cdot d\vec{r}$.*



Solution: By Stokes' Theorem,

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot d\vec{S}$$

$$= \iint_D (-Pz_x - Qz_y + R) dA,$$

where the surface S is $z = f(x, y) = 6 - 2x - 2y$, $(P, Q, R) = \nabla \times \vec{F}$.

By $z = 6 - 2x - 2y$, $z_x = -2$, $z_y = -2$.

$$\nabla \times \vec{F},\ or,\ curl\vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k} = 2\vec{i} + \vec{j} + 2x\vec{k}.$$

Thus

$$(P, Q, R) = (2, 1, 2x).$$

The projection D of S on the xy-plane is the disk $x^2 + y^2 \leq 4$. By polar coordinates, $x = r\cos\theta$, $y = r\sin\theta$,

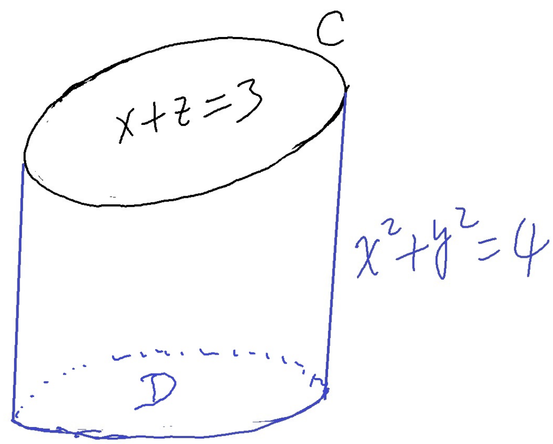
$$\int_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot d\vec{S}$$

$$= \iint_D (-Pz_x - Qz_y + R) dA$$

$$= \iint_D (6 + 2x)\ dA$$

$$= \int_0^2 \int_0^{2\pi} (6 + 2r\cos\theta) r dr d\theta = 24\pi.$$

Example 98. *Let $\vec{F} = (-3y^2, 2x, \sin(z^2+1))$, and let S be the surface whose boundary C is the curve of intersection of the plane $x + z = 3$ and the cylinder $x^2 + y^2 = 4$. Using Stokes' Thm evaluate $\int_C \vec{F} \cdot d\vec{r}$.*



Solution: By Stokes' Theorem,

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot d\vec{S}$$

$$= \iint_D (-Pz_x - Qz_y + R) dA,$$

where $\nabla \times \vec{F} = (P, Q, R)$. The surface S: $z = 3 - x$. Thus $z_x = -1$, $z_y = 0$.

$$\nabla \times \vec{F},\ or,\ curl\vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k} = (2 + 6y)\vec{k}.$$

Thus

$$(P, Q, R) = (0, 0, 2 + 6y).$$

The projection D of S on the xy-plane is the disk $x^2 + y^2 \leq 4$. By polar coordinates, $x = r\cos\theta$, $y = r\sin\theta$,

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot d\vec{S}$$

$$= \iint_D (-Pz_x - Qz_y + R)dA$$

$$= \iint_D (2 + 6y)\ dA$$

$$= \int_0^2 \int_0^{2\pi} (2 + 6r\sin\theta) r dr d\theta = 8\pi.$$

Example 99. *Let* $\vec{F} = z\vec{i} + 6x\vec{j} + 2y\vec{k}$, *and let* S *be the surface whose boundary* C *is the curve of intersection of the plane* $2x + 2y + z = 6$ *and the cylinder* $x^2 + y^2 = 4$. *Using Stokes' Thm find the flux of* $\nabla \times \vec{F}$, *i.e., evaluate* $\iint_S (\nabla \times \vec{F}) \cdot d\vec{S}$.

Solution: By Stokes' Theorem,

$$\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r}.$$

For the curve C, let $x = 2\cos t$, $y = 2\sin t$, then $z = 6 - 4\cos t - 4\sin t$, $0 \le t \le 2\pi$.

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot \vec{r}\,'(t) dt$$

$$= \int_0^{2\pi} (6 - 4\cos t - 4\sin t, 12\cos t, 4\sin t) \cdot (-2\sin t, 2\cos t, 4\sin t - 4\cos t) dt$$

$$= \int_0^{2\pi} (-12\sin t - 8\sin t\cos t + 24) dt$$

$$= (12\cos t - 4\sin^2 t + 24t)|_{t=0}^{2\pi} = 48\pi$$

6.5 Triple Integrals

Triple integral of $f(x, y, z)$ **over the solid** E**:**

$$\iiint_E f(x, y, z) dV.$$

- If $f = 1$, the triple integral is **the volume** of the solid E.
- If f is the density function, the triple integral gives **the mass** of E.
- If f is charge density, the triple integral gives **total charge** of E.

Triple Integral over a Rectangular Box

A rectangular box is the region in 3-dimensional space defined by

$$B = \{(x, y, z) : a \le x \le b, c \le y \le d, r \le z \le s\}.$$

Fubini's Theorem: If $f(x, y, z)$ is continuous on the rectangular box B, then

$$\iiint_B f(x, y, z)dV = \int_r^s \int_c^d \int_a^b f(x, y, z)dx\, dy\, dz.$$

This integral can also be evaluated by the other orders of the variables.

Example 100. *Find the mass of* $B = \{(x, y, z) : 0 \le x \le 2, 0 \le y \le 2, 0 \le z \le 2\}$ *with the density function*

$$\delta(x, y, z) = x + y + z.$$

Solution:

$$M = \iiint_B \delta(x, y, z)dV = \iiint_B (x + y + z)dV$$

$$= \int_0^2 \int_0^2 \int_0^2 (x + y + z)dx\, dy\, dz = \int_0^2 \int_0^2 \left(\frac{1}{2}x^2 + xy + xz)\right|_{x=0}^2 dy\, dz$$

$$= \int_0^2 \int_0^2 (2y + 2z + 2)\, dy\, dz = \int_0^2 \left.(y^2 + 2yz + 2y)\right|_{y=0}^2 dz = \int_0^2 (4z + 8)\, dz = 24.$$

Triple Integrals over a General Region

z**-simple region (Type I).** The region is bounded by a cylinder $F(x, y) = 0$, and the graphs of two functions of x and y:

$$E = \{(x, y, z) : (x, y) \in D, u_1(x, y) \le z \le u_2(x, y)\},$$

where D is the region in (x, y) plane bounded by the graph of $F(x, y) = 0$. A triple integral over a region E of type I is evaluated by

$$\iiint_E f(x, y, z)dV = \iint_D \left(\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)dz\right) dA.$$

x**-simple region (Type II).** The region is bounded by a cylinder $F(y, z) = 0$, and the graphs of two functions of y and z:

$$E = \{(x, y, z) : (y, z) \in D, u_1(y, z) \le x \le u_2(y, z)\},$$

where D is the region in (y, z) plane bounded by the graph of $F(y, z) = 0$. A triple integral over a region E of type II is evaluated by

$$\iiint_E f(x, y, z)dV = \iint_D \left(\int_{u_1(y,z)}^{u_2(y,z)} f(x, y, z)dx \right) dA.$$

y-simple region (Type III). The region is bounded by a cylinder $F(x, z) = 0$, and the graphs of two functions of x and z:

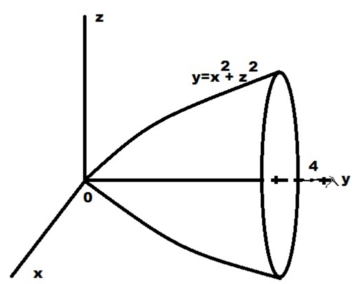
$$E = \{(x, y, z) : (x, z) \in D, u_1(x, z) \leq y \leq u_2(x, z)\},$$

where D is the region in (x, z) plane bounded by the graph of $F(x, z) = 0$. A triple integral over a region E of type III is evaluated by

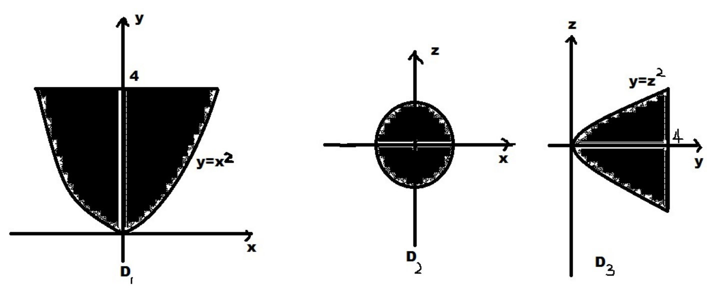
$$\iiint_E f(x, y, z)dV = \iint_D \left(\int_{u_1(x,z)}^{u_2(x,z)} f(x, y, z)dy \right) dA.$$

Remark. If a region is not of any of these types, we can subdivide this region into a finite number of regions of these types. The triple integral is the sum of triple integrals over sub-regions.

Example 101. *Describe the region E by z-simple, x-simple, and y-simple respectively, where E is the region bounded by paraboloid $y = x^2 + z^2$ and $y = 4$.*



The projections are:



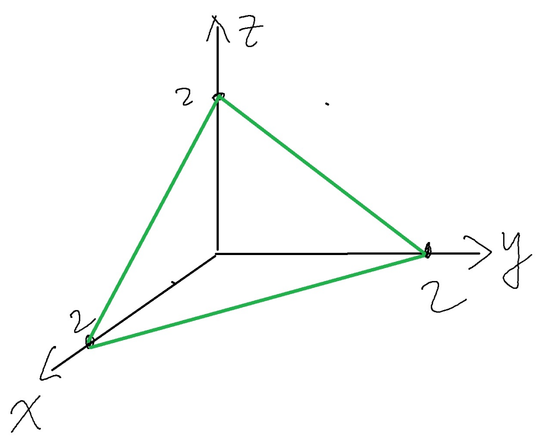
Solution:

- **z-simple:** $-\sqrt{y-x^2} \le z \le \sqrt{y-x^2}$, $x^2 \le y \le 4$, $-2 \le x \le 2$.

- **y-simple:** $x^2+z^2 \le y \le 4$, $-\sqrt{4-x^2} \le z \le \sqrt{4-x^2}$, $-2 \le x \le 2$.

- **x-simple:**$-\sqrt{y-z^2} \le x \le \sqrt{y-z^2}$, $z^2 \le y \le 4$, $-2 \le z \le 2$.

Example 102. *Electric charge is distributed on the solid E with charge density $\rho = x+y$, where E is the tetrahedron bounded by planes $x=0, y=0, z=0$, and $x+y+z=2$. Find the total charge.*



Solution: We consider E as z-simple region. Then the projection to xy-plane is the triangle bounded by $x=0$, $y=0$ and $x+y=2$. Thus

$$0 \le z \le 2-x-y, 0 \le y \le 2-x, 0 \le x \le 2.$$

$$\textit{total charge} = \iiint_E (x+y)dV = \int_0^2 \int_0^{2-x} \int_0^{2-x-y} (x+y)\ dzdydx$$

$$= \int_0^2 \int_0^{2-x} (x+y)(2-x-y)\ dydx$$

$$= \int_0^2 \int_0^{2-x} (2x-x^2-2xy+2y-y^2)\ dydx$$

$$= \int_0^2 \left[(2x-x^2)(2-x) - x(2-x)^2 + (2-x)^2 - \frac{1}{3}(2-x)^3\right]\ dx$$

$$= \int_0^2 \left[(2-x)^2 - \frac{1}{3}(2-x)^3\right]\ dx$$

$$= -\int_2^0 \left[u^2 - \frac{1}{3}u^3\right] dx, \qquad u = 2 - x$$

$$= (\frac{1}{3}u^3 - \frac{1}{12}u^4)|_0^2 = \frac{4}{3}.$$

6.6-6.7 Change of Variables in Triple Integrals

Suppose that we want to integrate $f(x, y, z)$ over the region R. Under the transformation $T : x = g(u, v, w), y = h(u, v, w), z = k(u, v, w)$ the region R becomes S, and the integral becomes,

$$\iiint_R f(x, y, z)dV = \iiint_S f(g, h, k)\frac{\partial(x, y, z)}{\partial(u, v, w)}dudvdw,$$

where

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

and is called the Jacobian of the transformation T, and

$$dV = \frac{\partial(x, y, z)}{\partial(u, v, w)}dudvdw.$$

- Cylindrical coordinates: $J(r, \theta, z) = r$.
- Spherical coordinates: $J(\rho, \phi, \theta) = \rho^2 \sin\phi$.

Triple Integrals in Cylindrical Coordinates

Cylindrical coordinate system uses (r, θ, z) to specify a point in space, where r and θ are polar coordinates of the projection of the point on the xy-plane.

$$x = r\cos\theta, y = r\sin\theta, z = z; \quad r = \sqrt{x^2 + y^2}, \cos\theta = \frac{x}{r}, \sin\theta = \frac{y}{r}, z = z.$$

Example 103. $(r, \theta, z) = (-1, \pi, 2) \Rightarrow (x, y, z) = (1, 0, 2)$; $(x, y, z) = (\sqrt{3}, -1, 2) \Rightarrow (r, \theta, z) = (2, -\pi/6, 2)$.

Suppose the region E of integration is of type I. A triple integral can be evaluated with the cylindrical coordinates:

$$x = r\cos\theta, y = r\sin\theta, z = z, \quad dV = rdzdrd\theta.$$

Let D be the projection of E onto the xy-plane. Then

$$\iiint_E f(x,y,z)dV = \iint_D \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\ dz\, dA$$

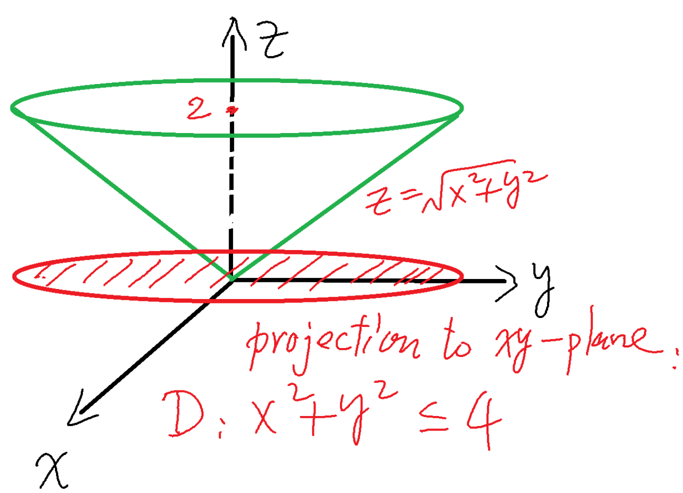
$$= \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z) r dz dr d\theta.$$

Example 104. *Find the integral* $\iiint_E (x^2+y^2)dV$ *, where E is the solid bounded by*

$$-2 \le x \le 2, -\sqrt{4-x^2} \le y \le \sqrt{4-x^2}, \sqrt{x^2+y^2} \le z \le 2.$$

Remark. The question is equivalent to the integral

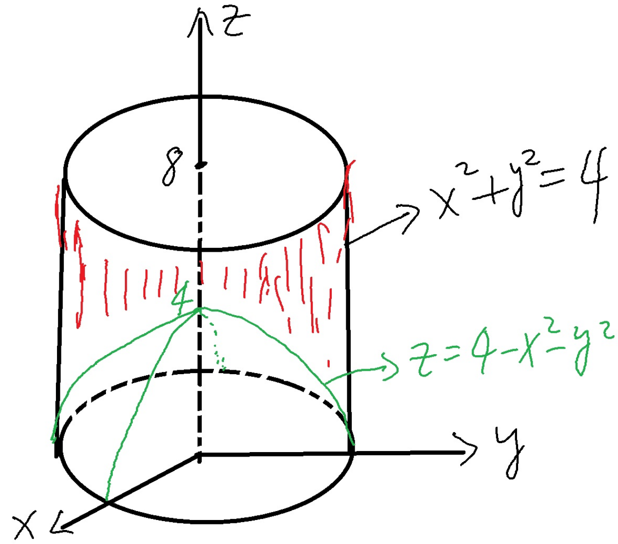
$$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{2} (x^2+y^2)\ dz\ dy\ dx.$$



Solution: Note that E is the region bounded by the cone $z = \sqrt{x^2+y^2}$ and the plane $z = 2$. In cylindrical coordinates, we obtain $0 \le r \le 2$, $0 \le \theta \le 2\pi$, $r \le z \le 2$. Thus

$$\iiint_E (x^2+y^2)dV = \int_0^{2\pi} \int_0^2 \int_r^2 r^2 r dz dr d\theta = \frac{16\pi}{5}.$$

Example 105. *Find the integral* $\iiint_E z dV$ *, where E is within* $x^2+y^2=4$*, below* $z=8$*, above* $z = 4-x^2-y^2$.



Solution: By cylindrical coordinates,

$$E = \{0 \le r \le 2, 0 \le \theta \le 2\pi, 4 - r^2 \le z \le 8\}.$$

$$\iiint_E z dV = \int_0^{2\pi} \int_0^2 \int_{4-r^2}^8 z \; r dz \; dr \; d\theta$$

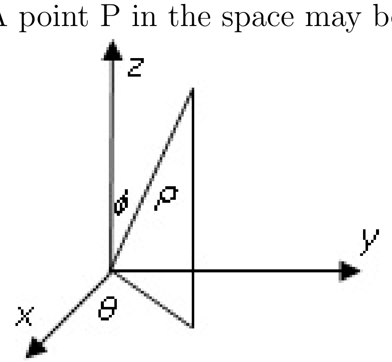
$$= \int_0^{2\pi} \int_0^2 \frac{1}{2} r z^2 |_{z=4-r^2}^8 \; dr \; d\theta$$

$$= \int_0^{2\pi} \int_0^2 \frac{1}{2} r(48 + 8r^2 - r^4) \; dr \; d\theta$$

$$= \int_0^{2\pi} \frac{1}{2} \left(24r^2 + 2r^4 - \frac{1}{6} r^6 \right) |_{r=0}^2 \; d\theta = \frac{352}{3}\pi$$

Triple Integrals in Spherical Coordinates

A point P in the space may be specified (ρ, θ, ϕ):



$$x = \rho \sin\phi \cos\theta, y = \rho \sin\phi \sin\theta, z = \rho \cos\phi, dV = \rho^2 \sin\phi d\rho d\phi d\theta,$$

where $\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = \rho^2 sin\phi$.

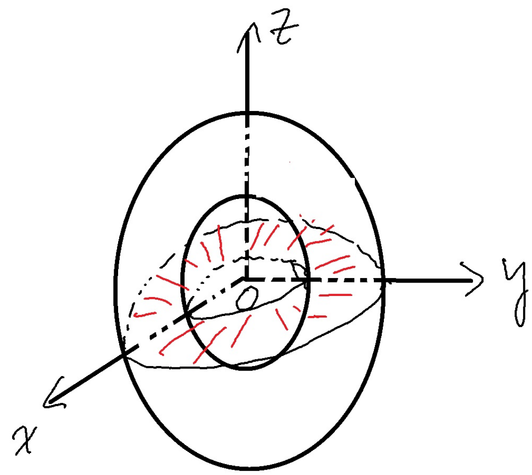
Example 106. *$(x, y, z) = (-\sqrt{3}, 1, 2)$, find (ρ, θ, ϕ).*

If a region E is specified in spherical coordinates, then

$$\iiint\limits_E f(x,y,z)dV = \iiint\limits_E f(\rho sin\phi \cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2\sin\phi d\rho d\phi d\theta,$$

where the order of integration depends on the definition of E.

Example 107. *Find the integral $I = \iiint\limits_E \sqrt{x^2+y^2+z^2}dV$, where E is region between the sphere $x^2+y^2+z^2=1$, and the sphere $x^2+y^2+z^2=4$.*

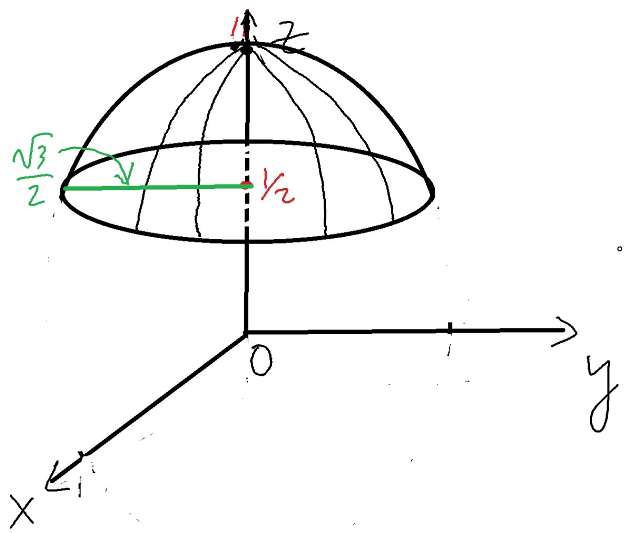


Solution: It is easy to see that $1 \le \rho \le 2, 0 \le \theta \le 2\pi, 0 \le \phi \le \pi$. Thus

$$I = \iiint\limits_E \sqrt{x^2+y^2+z^2}dV = \int_0^{2\pi}\int_0^{\pi}\int_1^2 \sqrt{\rho^2}\rho^2\sin\phi d\rho d\phi d\theta$$

$$= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi}\sin\phi\, d\phi\right)\left(\int_1^2 \rho^3 d\rho\right) = 15\pi.$$

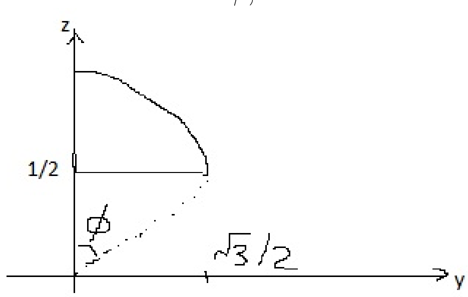
Example 108. *Find the integral $I = \iiint\limits_E zdV$, where E is region within the sphere $x^2+y^2+z^2=1$ and above the plane $z=1/2$.*



Solution: The intersection of the sphere $x^2 + y^2 + z^2 = 1$ and the plane $z = 1/2$ is

$$x^2 + y^2 = 3/4, \Rightarrow 0 \le \theta \le 2\pi.$$

To find the interval for ϕ, we look at the intersection of the solid and the yz-plane:



$$y^2 = 3/4 \Rightarrow y = \sqrt{3}/2 \Rightarrow \tan\phi = \frac{\sqrt{3}/2}{1/2} = \sqrt{3} \Rightarrow \phi = \pi/3 \Rightarrow 0 \le \phi \le \pi/3.$$

$$z \ge 1/2 \Rightarrow \rho\cos\phi \ge 1/2 \Rightarrow \rho \ge \frac{1}{2\cos\phi}.$$

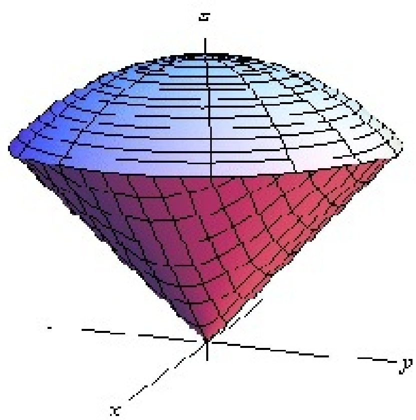
$$x^2 + y^2 + z^2 \le 1 \Rightarrow \rho^2 \le 1 \Rightarrow \rho \le 1.$$

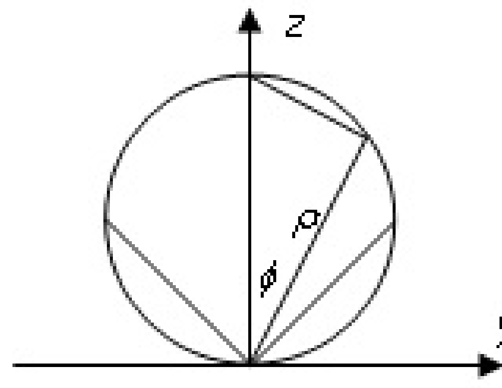
Thus

$$I = \iiint_E z dV = \int_0^{2\pi} \int_0^{\pi/3} \int_{1/(2\cos\phi)}^1 \rho\cos\phi\rho^2 \sin\phi d\rho d\phi d\theta$$

$$= \frac{1}{64} \int_0^{2\pi} \int_0^{\pi/3} \sin\phi\cos\phi \left(16 - \frac{1}{\cos^4\phi}\right) d\phi d\theta = \frac{9\pi}{64}.$$

Example 109. *Find the volume of the solid above the cone $z = \sqrt{x^2+y^2}$ and within the sphere $x^2+y^2+(z-1/2)^2 = 1/4$.*





Solution: The intersection of this solid and the yz-plane is shown above. We need to find ranges for ρ, θ, ϕ. Note that the solid is within the cone and below the sphere, we have

$$x^2+y^2-z^2 \leq 0, \Rightarrow \rho^2\sin^2\phi - \rho^2\cos^2\phi \leq 0, \Rightarrow \sin\phi \leq \cos\phi, \Rightarrow 0 \leq \phi \leq \frac{\pi}{4}.$$

$$x^2+y^2+(z-1/2)^2 \leq 1/4 \Rightarrow x^2+y^2+z^2-z \leq 0 \Rightarrow \rho^2 - \rho\cos\phi \leq 0 \Rightarrow 0 \leq \rho \leq \cos\phi.$$

The intersections of $x^2+y^2-z^2=0$ and $x^2+y^2+(z-1/2)^2 = 1/4$ are $z = 0, 1/2$, which gives the domain in the xy-plane: $x^2+y^2 = 1/4$. Thus $0 \leq \theta \leq 2\pi$.

$$V = \iiint\limits_E dV = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{\cos\phi} \rho^2 \sin\phi d\rho d\phi d\theta = \int_0^{2\pi}\int_0^{\pi/4} \frac{1}{3}\cos^3\phi\sin\phi d\phi d\theta = \frac{\pi}{8}.$$

6.8 The Divergence Theorem

The divergence theorem relates a surface integral to a triple integral.

Let E be a simple solid region bounded by a closed piecewise-smooth surface S, and let $\vec{n}$ be the unit outer normal to S. If $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ is a vector field, where P, Q, R have continuous partial derivatives on an open region containing E, then

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_E (div\vec{F})dV.$$

Example 110. *Evaluate* $\iint_S \vec{F} \cdot d\vec{S}$*, where* $\vec{F} = 3xz^2\vec{i} + 3y\vec{j} - z^3\vec{k}$*,* S *is the unit sphere* $x^2 + y^2 + z^2 = 1$*.*

Solution:

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E div\vec{F} dV = \iiint_E 3dV = 3 \cdot \frac{4}{3}\pi(1)^3 = 4\pi.$$

Example 111. *Evaluate* $\iint_S \vec{F} \cdot d\vec{S}$*, where* $\vec{F} = 3xz^2\vec{i} + (3y + yz)\vec{j} - z^3\vec{k}$*,* S *is the unit sphere* $x^2 + y^2 + z^2 = 1$*.*

Solution:

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E div\vec{F} dV = \iiint_E (3 + z) dV$$

$$= \int_0^{2\pi} \int_0^{\pi} \int_0^1 (3 + \rho\cos\phi)\ \rho^2 \sin\phi d\rho d\phi d\theta = \int_0^{2\pi} \int_0^{\pi} \left. (\rho^3 + \frac{1}{4}\rho^4\cos\phi) \right|_{\rho=0}^{1} \sin\phi d\phi d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi} (\sin\phi + \frac{1}{4}\cos\phi\sin\phi) d\phi d\theta = \int_0^{2\pi} (-\cos\phi + \frac{1}{8}\sin^2\phi)|_{\phi=0}^{\pi}\ d\theta$$

$$= \int_0^{2\pi} (2)\ d\theta = 4\pi.$$

Example 112. *Evaluate* $\iint_S \vec{F} \cdot d\vec{S}$*, where* $\vec{F} = 2xy^2\vec{i} + 2x^2y\vec{j} + z\vec{k}$*,* S *consists of three surfaces:* $z = 4 - 3x^2 - 3y^2$*,* $1 \le z \le 4$ *on the top;* $x^2 + y^2 = 1$*,* $0 \le z \le 1$ *on the side;* $z = 0$ *on the bottom.*

Solution:

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E div\vec{F} dV = \iiint_E (2x^2 + 2y^2 + 1) dV.$$

$$= \int_0^{2\pi} \int_0^1 \int_0^{4-3r^2} (2r^2 + 1)\ r dz dr d\theta = 2\pi \int_0^1 (4 - 3r^2)(2r^3 + r)\ dr$$

$$= \frac{9\pi}{2}.$$

6.11 Maxima and Minima

Definition 3. *We say that a function* $f(x, y)$ *has a relative (local) maximum at a point* (x_0, y_0) *if there is a circle centered at* (x_0, y_0) *such that*

$$f(x, y) \ge f(x_0, y_0)$$

for all (x, y) in that circle; $f(x, y)$ has a relative (local) minimum at a point (x_0, y_0) if there is a circle centered at (x_0, y_0) such that

$$f(x, y) \leq f(x_0, y_0)$$

for all (x, y) in that circle.

Definition 4. *The critical points of a function $f(x, y)$ are those points (x_0, y_0) for which $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$, or if $f_x(x_0, y_0)$ or $f_y(x_0, y_0)$ is undefined. Saddle point: The graph of the function crosses the tangent plane at this point.*

First-Partials Test for Relative Extrema: If f has a relative extrema at (a, b), and the first partial derivatives exist in a circle centered at (a, b), then (a, b) is a critical point.

Second-Partials Test for Relative Extrema: Assume that f has a continuous partial derivatives on an open region containing (a, b). Let (a, b) be a critical point of f. Denote

$$d(a, b) = f_{xx}(a, b) f_{yy}(a, b) - (f_{xy}(a, b))^2 .$$

1. If $d > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a relative minimum.
2. If $d > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a relative maximum.
3. If $d < 0$, then $f(a, b)$ has a saddle point.
4. If $d = 0$, then the second derivatives test gives nothing.

Example 113. *Classify the critical points of $f(x, y) = x^3 - 3xy + y^3$*

Solution:

$$f_x(x, y) = 3x^2 - 3y, \quad f_y(x, y) = -3x + 3y^2, \Rightarrow$$

$$f_{xx}(x, y) = 6x, \quad f_{yy}(x, y) = 6y, \quad f_{xy}(x, y) = -3.$$

Setting $f_x = 0$ and $f_y = 0 :\ 3x^2 - 3y = 0,\ -3x + 3y^2 = 0.$ We imply that $(x, y) = (0, 0), (1, 1)$.

$$d(x, y) = f_{xx}(x, y) f_{yy}(x, y) - (f_{xy}(x, y))^2 = 36xy - 9.$$

- $d(0, 0) = -9$, so $(0, 0)$ is a saddle point.
- $d(1, 1) = 27,\ f_{xx}(1, 1) = 6 > 0$, so $f(1, 1)$ is a relative minimum.

Example 114. *Find the critical point(s) of $f(x, y) = x(x - 2)y(y + 4)$ and classify them.*

Solution:

$$f_x = (2x-2)y(y+4), \quad f_y = x(x-2)(2y+4).$$

Set

$$f_x = 0, \quad f_y = 0,$$

i.e.,

$$(2x-2)y(y+4) = 0, \quad x(x-2)(2y+4) = 0.$$

So critical points are

$$(1,-2), (0,0), (0,-4), (2,0), (2,-4).$$

To test all of them, we use the Second-Partials Test.

$$f_{xx}(x,y) = 2y(y+4), \quad f_{yy}(x,y) = 2x(x-2), \quad f_{xy}(x,y) = (2x-2)(2y+4).$$

$$d(x,y) = f_{xx}(x,y)f_{yy}(x,y) - (f_{xy}(x,y))^2.$$

- $d(1,-2) = 16 > 0$, $f_{xx}(1,-2) = -2 < 0$, so $f(1,-2)$ is a relative max.
- $d(0,0) = -64 < 0$, so (0,0) is a saddle point.
- $d(0,-4) = -64 < 0$, so (0,-4) is a saddle point.
- $d(2,0) = -64 < 0$, so (2,0) is a saddle point.
- $d(2,-4) = -64 < 0$, so (2,-4) is a saddle point.

6.12 Lagrange Multipliers

Case 1. Two variables with one constraint: Find the extreme values of the function $z = f(x,y)$ subject to constraint $g(x,y) = 0$.

Interpretation. Find extreme values of $f(x,y)$ on a curve.

When z attains an extreme value a, the level curve $f(x,y) = a$ and $g(x,y) = 0$ have the same tangent line. Hence their gradient vectors have the same or opposite direction: $\nabla f = \lambda \nabla g$, λ is a constant, called Lagrange Multiplier.

Method of Lagrange Multipliers:

- Solve the system of equations:

$$\nabla f = \lambda \nabla g, \quad g(x,y) = 0.$$

  Let's say, the solutions are (x_i, y_i), $i = 1, ..., n$.

- The maximum $= max\{f(x_i, y_i): \ i = 1, ..., n\}$. The minimum $= min\{f(x_i, y_i): \ i = 1, ..., n\}$.

Example 115. *Find the max and min of $z = f(x, y) = xy$, subject to $x^2 + y^2 = 1$.*

Solution: Here $g(x, y) = x^2 + y^2 - 1$. $\nabla f = (y, x), \nabla g = (2x, 2y)$. So we have

$$\begin{aligned} y &= \lambda 2x, \\ x &= \lambda 2y, \\ x^2 + y^2 &= 1. \end{aligned}$$

We have $x = \pm\frac{1}{\sqrt{2}}$, $y = \pm\frac{1}{\sqrt{2}}$.

The maximum value of z is $z(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = z(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) = 1/2$,

and the minimum value of z is $z(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) = z(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = -1/2$.

Case 2. Three variables with one constraint: Find the extreme values of the function $w = f(x, y, z)$ subject to constraint $g(x, y, z) = 0$.

Interpretation. Find the extreme values of (x, y, z) on a surface.

When w attains an extreme value a, the level curve $f(x, y, z) = a$ and $g(x, y, z) = 0$ have the same tangent plane. Hence their gradient vectors have the same or opposite direction: $\nabla f = \lambda \nabla g$, λ is a constant, called Lagrange Multiplier.

Method of Lagrange Multipliers:

- Solve the system of equations:

$$\nabla f = \lambda \nabla g, \quad g(x, y, z) = 0.$$

  Let's say, the solutions are (x_i, y_i, z_i), $i = 1, ..., n$.

- The maximum $= max\{f(x_i, y_i, z_i): \ i = 1, ..., n\}$. The minimum $= min\{f(x_i, y_i, z_i): i = 1, ..., n\}$.

Example 116. *Find the maximum and minimum of $f(x, y, z) = x + y + 2z$ subject to $x^2 + y^2 + z^2 = 6$.*

Solution: Here $g(x, y, z) = x^2 + y^2 + z^2 - 6$. $\nabla f = (1, 1, 2), \nabla g = (2x, 2y, 2z)$. By

$$\nabla f = \lambda \nabla g,$$

we have

$$\begin{aligned} 1 &= \lambda(2x), \\ 1 &= \lambda(2y), \\ 2 &= \lambda(2z), \\ x^2+y^2+z^2 &= 6. \end{aligned}$$

which gives $\lambda = \pm 1/2$. Thus

$$(x, y, z) = (1, 1, 2), \quad (-1, -1, -2).$$

The maximum value is $f(1, 1, 2) = 6$;
The minimum value is $f(-1, -1, -2) = -6$.

Example 117. *Find the maximum and minimum of $f(x, y, z) = xyz$ subject to $x+y+z = 1$.*

Solution: Here $g(x, y, z) = x + y + z - 1$. $\nabla f = (yz, xz, xy), \nabla g = (1, 1, 1)$. By

$$\nabla f = \lambda \nabla g, \quad g = 0,$$

we imply that

$$\begin{aligned} yz &= \lambda, && (1) \\ xz &= \lambda, && (2) \\ xy &= \lambda, && (3) \\ x+y+z-1 &= 0. && (4) \end{aligned}$$

(1)-(2): $z(y - x) = 0$, which implies that $z = 0$ or $x = y$;
(3)-(2): $x(y - z) = 0$, which implies that $x = 0$ or $z = y$.
Next we consider 4 combinations.

- If $z = 0, x = 0$: by (4), $y = 1$, so we get a point (0,1,0);
- If $z = 0, z = y$: then $y = 0$, by (4), $x = 1$, so we get a point (1,0,0);
- If $x = y, x = 0$: then $y = 0$, by (4), $z = 1$, so we get a point (0,0,1);
- If $x = y, z = y$: then by (4), $y = 1/3$, so we get a point (1/3,1/3, 1/3).

$f(1/3, 1/3, 1/3) = 1/27$, which is the maximum;
$f(1, 0, 0) = f(0, 0, 1) = f(0, 1, 0) = 0$, which is the minimum.

Example 118. *Find the maximum and minimum of* $f(x, y, z) = xyz$ *subject to* $2xz+2yz+xy-12 = 0$.

Solution: Here $g(x, y, z) = 2xz+2yz+xy-12$. $\nabla f = (yz, xz, xy)$, $\nabla g = (2z+y, 2z+x, 2x+2y)$. So we have

$$\begin{aligned} yz &= \lambda(2z+y), \\ xz &= \lambda(2z+x), \\ xy &= \lambda(2x+2y), \\ 2xz+2yz+xy-12 &= 0. \end{aligned}$$

We have $(x, y, z) = (2, 2, 1), (-2, -2, -1)$.
The maximum value of w is $w = f(2, 2, 1) = 4$;
The minimum value of w is $w = f(-2, 2, -1) = -4$.

Case 3. Three variables with two constraints: Find the extreme values of the function $w = f(x, y, z)$ subject to constraints $g(x, y, z) = 0$ and $h(x, y, z) = 0$.

Interpretation. Find extreme values of $f(x, y, z)$ on a 3-D curve.

When w attains an extreme value a, the tangent lines of the curve $g(x, y, z) = 0, h(x, y, z) = 0$, are in the tangent plane of the level surface $f(x, y, z) = a$. Hence, the gradient vector of level surface $f(x, y, z) = a$ and the gradient vectors of $g(x, y, z) = 0, h(x, y, z) = 0$ are in the same plane:

$$\nabla f = \lambda \nabla g + \mu \nabla h,$$

λ and μ are a constants.

Method of Lagrange Multipliers: Solve the system of equations:

$$\nabla f = \lambda \nabla g + \mu \nabla h, \quad g(x, y, z) = 0, h(x, y, z) = 0.$$

For each solution (x, y, z, λ, μ) of this system of equations, find the value of $f(x, y, z)$. The maximum is the maximum value of w, and the minimum is the minimum of w.
Remark. λ, μ are called Lagrange Multipliers.

Example 119. *Find the max and min of* $w = f(x, y, z) = x + y + 7z$ *subject to* $x - y + z = 1, x^2 + y^2 = 1$.

Solution: Here $g(x, y, z) = x - y + z - 1$, $h(x, y, z) = x^2 + y^2 - 1$,, $\nabla f = (1, 1, 7), \nabla g = (1, -1, 1), \nabla h = (2x, 2y, 0)$. So we have

$$\begin{aligned} 1 &= \lambda + \mu 2x, \\ 1 &= -\lambda + \mu 2y, \\ 7 &= \lambda, \\ x - y + z - 1 &= 0, \\ x^2 + y^2 - 1 &= 0. \end{aligned}$$

We have $(x, y, z) = (-0.6, 0.8, 2.4), (0.6, -0.8, -0.4)$.

The maximum value of w is $w = f(-0.6, 0.8, 2.4) = 17$;

and the minimum value of w is $w = f(0.6, -0.8, -0.4) = -3$.