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1. If $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ and B is a $3 \times n$ matrix then the second row of the matrix AB is

A. the same as the second row of B.
B. the sum of the first and second rows of B.
C. the sum of the second and third rows of B.
D. the sum of the first and third rows of B.
E. the same as the first row of A.
F. the same as the third row of A.

Write $B = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}$ in block row form; ie, r_i = ith row of B.

Then $AB = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} r_1 + r_3 \\ r_2 + r_3 \\ r_3 \\ 0 \end{bmatrix}$. Hence C is correct

2. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & -1 & 1 \end{bmatrix}$. What is the second row vector of A^{-1}?

A. $(-3, 1, 1)$
B. $(5, -3, -11)$
C. $(-1, 1, 0)$
D. $(1, \frac{1}{2}, 1)$
E. $(0, 1, 0)$
F. The matrix A is not invertible.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 2 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & -3 & -2 & -2 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & -2 & -2 & 3 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -\frac{3}{2} & -\frac{1}{2} \end{array}\right]$$

Further operations will not change the 2nd row, so we can stop! Hence C is correct

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3 . Find the value(s) of t for which $(2, 6, 5, 2t)$ lies in the subspace of $\mathbf{R}^4$ spanned by $(1, 2, 2, 2)$, $(3, 7, 6, 6)$ and $(1, 2, 1, 2)$.

A. $t = -4$ only.

B. $t = -2$ or -4.

C. $t = 0$ or 2.

D. $t = -2, 0$ or 4.

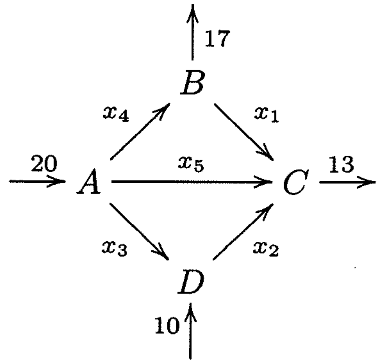
E. $t = 2$ or 4.

F. $t = 2$ only.

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 2 \\ 2 & 7 & 2 & 6 \\ 2 & 6 & 1 & 5 \\ 2 & 6 & 2 & 2t \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 1 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 2t-4 \end{array}\right]$$

This system is consistent $\Leftrightarrow$ $(2, 6, 5, 2t)$ is a l.c. of the columns $\Leftrightarrow$ $2t - 4 = 0 \Leftrightarrow t = 2$.

4. Consider the network of streets with intersections A, B, C and D below. The arrows indicate the direction of traffic flow along the one-way streets, and the numbers refer to the exact number of cars observed to enter or leave A B,C and D during one minute. Each x_i denotes the unknown number of cars which passed along the indicated streets during the same period.



[2] a) Write down a system of linear equations which describes the the traffic flow, **together with all the constraints** on the variables x_i, $i = 1, \ldots, 4$. (*Do not simply copy out the equations implicit in (b). You will not get any marks if you do this. Do not perform any operations on your equations: this is done for you in (b)!*)

Intersection	Flow in	=	Flow Out
A	20	=	$x_3 + x_4 + x_5$
B	x_4	=	$17 + x_1$
C	$x_1 + x_5 + x_2$	=	13
D	$x_3 + 10$	=	x_2

4 @ 1/4

Constraints (1/2) ① $x_i \geq 0$, $i = 1, \ldots, 5$ (one-way streets)

(1/2) ② $x_i \in \mathbb{Z}$, $i = 1, \ldots, 5$ (whole #'s of cars)

3/4

(Question 4 continued)

[1] b) The reduced row-echelon form of the augmented matrix from part (a) is

$$\begin{array}{c} \quad\quad\quad\quad s \quad t \\ \left[\begin{array}{ccccc|c} 1 & 0 & 0 & -1 & 0 & -17 \\ 0 & 1 & 0 & 1 & 1 & 30 \\ 0 & 0 & 1 & 1 & 1 & 20 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

Give the general solution. (Ignore the constraints at this point.)

$$\begin{aligned} x_1 &= -17 + s \\ x_2 &= 30 - s - t \\ x_3 &= 20 - s - t \quad ; \quad s, t \in \mathbb{R} \\ x_4 &= s \\ x_5 &= t \end{aligned}$$

5 @ $\frac{1}{5}$

[3] c) If $\overline{AC}$ were closed due to roadwork, using your results from (b), find

(i) The maximum flow along $\overline{DC}$, and $\frac{1}{2}$ - correct + 1 just^n

(ii) The minimum flow along $\overline{DC}$. $\frac{1}{2}$ - correct + 1 just^n

$\overline{AC}$ closed $\Leftrightarrow$ $x_5 = 0$ $\Leftrightarrow$ $t = 0$. ; Flow along $\overline{DC}$ is x_2.

Constraints

$$\left.\begin{aligned} x_1 \geq 0 &\Leftrightarrow -17 + s \geq 0 \Leftrightarrow s \geq 17 \\ x_2 \geq 0 &\Leftrightarrow 30 - s \geq 0 \Leftrightarrow 30 \geq s \\ x_3 \geq 0 &\Leftrightarrow 20 - s \geq 0 \Leftrightarrow 20 \geq s \\ x_4 \geq 0 &\Leftrightarrow s \geq 0 \end{aligned}\right\} *$$

$x_5 \geq 0 \Leftrightarrow t \geq 0$ (but $t = 0$.)

To satisfy all of $*$, we need

$$20 \geq s \geq 17$$

Hence $13 \geq 30 - s \geq 10$. Thus (i) 13 (ii) 10.

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5. Let $A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

$\boxed{2}$ a) Find the reduced row echelon form of A.

$$A \sim \underbrace{\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{RE}\checkmark} \sim \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is the RRE form of A

1 - Correct

1 - some just^n.

Give 1 if incorrect but their answer is in RRE form.

$\boxed{1\frac{1}{2}}$ b) Find a basis for $\ker A = \{x \in \mathbf{R}^4 \mid Ax = 0\}$. From A,

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \qquad \left.\begin{aligned} x_1 &= -s \\ x_2 &= -s \\ x_3 &= 0 \\ x_4 &= s \end{aligned}\right\} ;\ s \in \mathbb{R}$$

1 - Correct (or consistent with (a))

1/2 Justification ("basic sol^n", or any valid argument.)

Hence there is only one basic sol^n, namely $(-1,-1,0,1)$ and so $\{(-1,-1,0,1)\}$ is a basis for $\ker A$.

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(Question 5 continued)

[1] c) Is A invertible? (You do not need to find the inverse, if it exists.)

No, since rank $A = 3 < 4$ (by (a).)

(OR: No, since $\ker A \neq \{0\}$).

$\frac{1}{2}$ - correct

$\frac{1}{2}$ - justification

d) Extend your basis of $\ker A$ to a basis of $\mathbf{R}^4$, if necessary. (Be sure you justify your answer.)

We need v_2, v_3, v_4 so that $\begin{bmatrix} -1 & -1 & 0 & 1 \\ & v_2 & & \\ & v_3 & & \\ & v_4 & & \end{bmatrix}$

has rank 4. Clearly, $\begin{bmatrix} -1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ has

$\frac{1}{2}$ - any correct extension

1 - justification

rank 4, and so

$\{(-1,-1,0,1), (0,1,0,0), (0,0,1,0), (0,0,0,1)\}$

is a basis of $\mathbb{R}^4$ extending our basis of $\ker A$.

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6. Let A and B denote matrices, not necessarily square, and which have more than 1 row and more than 1 column, and let x denote a column vector (i.e., a $k \times 1$ matrix for some k).

State whether each of the following is (always) true, or is (possibly) false, in the box after the statement.

- If you say the statement may be false, you **must give an explicit example - with numbers!** *(Hint: Try an example with 2 or 3 rows or columns.)*
- If you say the statement is true, you must give a clear explanation - by quoting a theorem presented in class, any by giving other valid proof.

2 a) If A is $m \times n$ and rank $A = m$, then the system $Ax = 0$ has a unique solution. $\boxed{\text{FALSE}}$

e.g. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ Then since $2 = m < n = 3$, the homogeneous system $Ax = 0$ has ∞ly many solutions (rank $A = 2 < 3 = \#$variables)

1 correct + 1 justification (a, b, c).

2 b) If $AB = 0$ then either $A = 0$ or $B = 0$. $\boxed{\text{FALSE}}$

e.g. Let $A = B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then

$$AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$ but

neither A nor B is zero.

2 c) If B has a column of zeros then AB has a column of zeros. $\boxed{\text{TRUE}}$

Write $B = [c_1\ c_2 \ldots \underset{}{\overset{j^{th}}{0}} \cdots c_n]$. Then

$$AB = [Ac_1\ Ac_2 \ldots A.0 \ldots Ac_n] = [Ac_1\ Ac_2 \ldots 0 \ldots Ac_n].$$

i.e. if the j^{th} col. of B is zero, the j^{th} col. of AB is zero as well, as $A(j^{th}$ col of $B) = j^{th}$ col of (AB)

7. [Bonus] Suppose A is an invertible 5×5 matrix and B is any 5×4 matrix with rank $B = 4$. Prove carefully that rank $AB = 4$.

We know that for an $m \times \underline{n}$ matrix C,

$$\text{rank } C = \underline{n} \iff (Cx = 0 \Rightarrow x = 0).$$

So suppose $(AB)x = 0$. Since A is invertible,

$$A^{-1}(AB)x = A^{-1}0 = 0$$

ie $(A^{-1}A)Bx = 0$

ie. $Bx = 0$. $(*)$

But rank $B = 4$ and B is a $5 \times \underline{4}$ matrix, so $(*)$ implies $x = 0$. Hence,

$$(AB)x = 0 \Rightarrow x = 0.$$

Thus the rank of AB = # cols of AB $= \underline{4}$.

1 - Some idea + some progress

+ 1 " + " + almost perfect

+ 1 correct and well-written.