1. Suppose $\{u, v, w\}$ is a set of vectors in a vector space V. Which of the following statements are equivalent to

"$\{u, v, w\}$ is linearly independent."

I. None of the vectors u, v or w is a multiple of any other single vector in $\{u, v, w\}$. ✗

II. If a, b, c are scalars, then $au + bv + cw = 0$ implies $a = b = c = 0$. ✓

III. Both $\{u, v\}$ and $\{v, w\}$ are linearly independent. ✗

IV. If $a = b = c = 0$, then $au + bv + cw = 0$. (✗ – this is true for any vectors u, v, w.)

V. None of the vectors u, v or w is a linear combination of the other vectors in $\{u, v, w\}$. ✓

A. I & II

B. I & III

C. II & III

D. II & V

E. II & IV

F. III & IV

I: e.g. $\{\underset{u}{(1,0,0)}, \underset{v}{(0,1,0)}, \underset{w}{(1,1,0)}\}$ satisfies I but is l.d.

III: See the example in I

2. Let $X = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \in \mathbf{M}_{22} \mid a - d = 0 \right\}$. Which of the following is a spanning set for X?

$X = \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \middle| \, a, b \in \mathbb{R} \right\}$

$= \operatorname{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}.$

Hence D is correct (and no others are.)

A. $\left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$

B. $\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\}$

C. $\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

D. $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$

E. $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

F. $\left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$

3. Which of the following are subspaces of $\mathbf{M}_{22}$?

$U = \left\{ \begin{bmatrix} x & y \\ y & z \end{bmatrix} \in \mathbf{M}_{22} \mid x,\ y,\ z \in \mathbf{R} \right\}$ $= \text{span} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ ✓

$V = \left\{ \begin{bmatrix} 0 & y \\ -y & 0 \end{bmatrix} \in \mathbf{M}_{22} \mid y \in \mathbf{R} \right\}$ $= \text{span} \left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$ ✓

$W = \left\{ \begin{bmatrix} x & y \\ z & w \end{bmatrix} \in \mathbf{M}_{22} \mid xw - zy = 0 \right\}$ ✗ (see below)

$S = \left\{ \begin{bmatrix} x & y \\ z & -x \end{bmatrix} \in \mathbf{M}_{22} \mid x,\ y,\ z \in \mathbf{R} \right\}$ $= \text{span} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$

A. Only U and V
B. Only U and W
C. Only U, V and W
D. Only V, W and S
E. Only W and S
F. Only U, V and S

W is not a subspace because

$M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in W$ and $M_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in W$

but $M_1 + M_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \notin W$.

(i.e. W is not closed under addn.)

(We have used the fact that the span of any set of vectors is always a subspace. See your class notes, or the text, Thm 4 P. 299)

4. Let $n_0 = (1,0,1) \in \mathbf{R}^3$ and define

$$U = \{v \in \mathbf{R}^3 \mid ||v - n_0|| = ||v + n_0||\}.$$

($\frac{1}{2}$) a) If $v \in \mathbf{R}^3$, show by expanding both sides using the dot product that

$$||v - n_0||^2 = ||v + n_0||^2 \iff v \cdot n_0 = 0.$$

($\star$) Now we know that $U = \{v \in \mathbf{R}^3 \mid v \cdot n_0 = 0\}$, and thus U is a subspace of $\mathbf{R}^3$.

($1\frac{1}{2}$) b) Give a complete geometric description of U. (Hint: use ($\star$)!)

(2) c) Find a spanning set for U.

(2) d) Is your spanning set in (d) linearly independent?

a) $||v - n_0||^2 = (v - n_0)\cdot(v - n_0) = ||v||^2 - 2v\cdot n_0 + ||n_0||^2$, while a similar computation shows that $||v + n_0||^2 = ||v||^2 + 2v\cdot n_0 + ||n_0||^2$. Hence $||v - n_0||^2 = ||v + n_0||^2 \iff -2v\cdot n_0 = 2v\cdot n_0 \iff v\cdot n_0 = 0$.

b) By ($\star$), U is the plane (1/2) through the origin (1/2) with normal vector $n_0 = (1,0,1)$ (1/2)

c) $U = \{(x,y,z) \in \mathbb{R}^3 \mid x + z = 0\} = \{(x,y,-x) \mid x,y \in \mathbb{R}\}$
$= \text{span}\{(1,0,-1), (0,1,0)\}$. Hence $\{\underset{v_1}{(1,0,-1)}, \underset{v_2}{(0,1,0)}\}$ is a spanning set for U. (1) - any correct spanning set
(1) - just^n

d) Yes, because (i) neither of the 2 vectors v_1, v_2 above is a multiple of the other

or (ii) $av_1 + bv_2 = 0 \Rightarrow (a, b, -a) = (0,0,0)$
$\Rightarrow a = b = 0$.

(1) - correct answer
(1) - correct just^n

5. Let $\mathbf{F}[0,\pi] = \{f \mid f : [0,\pi] \to \mathbf{R}\}$ be the vector space of real-valued functions defined on $[0,\pi]$. Define four functions in $\mathbf{F}([0,\pi])$ by

$$f(x) = 1, \quad g(x) = \cos 2x, \quad h(x) = \cos x, \quad \text{and} \quad k(x) = \sin^2 x, \qquad \forall x \in [0,\pi],$$

and let $W = \text{span}\{f, g\}$.

2 a) Show that $\{f, g\}$ is linearly independent.

2 b) Show that $h \notin W$.

c) Use trigonometric identities to show that $k \in W$. (*Hint:* $\cos 2x = \cos(x + x)$.)

a) Suppose $a, b \in \mathbb{R}$ and $af + bg = 0$. (*) Then

$a + b.\cos 2x = 0, \quad \forall x \in [0,\pi]$.

For $x = 0$, this yields $a + b = 0$

& $x = \pi/4$ " $a + 0 = 0$

Together, ~~these~~ which imply that $a = b = 0$.

$\therefore \{f, g\}$ is l.i..

0

1 - knowing + what to do

1 - any system consistent with (*) which implies $a = b = 0$

b) Suppose $h = af + bg$ for some $a, b \in \mathbb{R}$. (**)

Then $\cos x = a + b\cos 2x$, $\forall x \in [0,\pi]$.

1 - setting it up

1 any system consistent with (**) that is inconsistent

At $x = 0$, we obtain $1 = a + b$

$x = \frac{\pi}{2}$ " $0 = a - b$

$x = \pi$ " $-1 = a + b$

– These 2 equations have no sol^n for a & b.

Hence (**) is impossible. Thus $h \notin \text{span}\{f, g\} = W$

0

c) Note that $\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x$ $(\forall x \in \mathbb{R})$

Hence $g = 1 - 2k$, or $g = f - 2k$, so

$k = \frac{1}{2}f - \frac{1}{2}g \in \text{span}\{f, g\} = W$.

1 knowing what to do

1 correct sol^n

6. [Bonus] Let U and V be subspaces of a vector space W such that the subset

$$U \cup V = \{w \in W \mid w \in U \text{ or } w \in V\}$$ Either 1 or

is also a subspace of W. 2 marks. close to

Prove carefully that either $U \cup V = U$ or $U \cup V = V$. 1 - very good 2 - perfect

(You may find the following fact useful: If U is not a subset of V and V is not a subset of U, then there exists $u \in U$ with $u \notin V$ and $v \in V$ with $v \notin U$.)

Suppose neither $U \cup V = U$ or $U \cup V = V$. Then $U \not\subseteq V$ and $V \not\subseteq U$. (*) Hence $\exists u \in U$ with $u \notin V$ and $v \in V$ with $v \notin U$. (**)

Consider $w = u + v$. Since $U \cup V$ is a subspace, and $u \in W$ and $v \in W$, we know $u + v \in U \cup V$.

Hence either (I) $u + v \in U$

or (II) $u + v \in V$.

If (I), then $u + v = u'$ for some $u' \in U$, so $v = u' - u$. As U is a subspace, and $u, u' \in U$, $v = u' - u \in U$, a contradiction to (**). Thus (I) cannot occur.

If (II), then $u + v = v'$ for some $v' \in V$, so $u = v' - v$. Again, V is also a subspace and so $u = v' - v \in V$, contradicting (**) again.

Neither (I) nor (II) is possible, so our initial assumption (*) is false. Thus either $U \cup V = U$ or $U \cup V = V$