Final F13 Solns

1. The subset $\{(x, y, z) \in \mathbf{R}^3 \mid 5x - 6y = -1\}$ is

A. a line in $\mathbf{R}^2$ with direction vector $(5,\ -6)$.
B. a line in $\mathbf{R}^2$ with direction vector $(-6,\ 5)$.
C. a plane in $\mathbf{R}^3$ through $(1,\ 1,\ 1)$ with normal vector $(5,\ -6,\ 0)$.
D. a plane in $\mathbf{R}^3$ through $(1,\ 1,\ 1)$ with normal vector $(-6,\ 5,\ 0)$.
E. a line in $\mathbf{R}^3$ with direction vector $(5,\ -6)$.
F. a line in $\mathbf{R}^3$ with direction vector $(-6,\ 5)$.

2. Which two of the following are subspaces of $\mathbf{F}(\mathbf{R}) = \{f \mid f : \mathbf{R} \to \mathbf{R}\}$?

$S = \{f \in \mathbf{F}(\mathbf{R}) \mid f(-1)f(1) = 0\}$ ✗
$T = \{f \in \mathbf{F}(\mathbf{R}) \mid f(-1) = f(1)\}$ ✓
$U = \{f \in \mathbf{F}(\mathbf{R}) \mid f(1) = 1\}$ ✗
$V = \{f \in \mathbf{F}(\mathbf{R}) \mid f(0) = 0\}$ ✓

A. S and T.
B. S and U.
C. S and V.
D. T and U.
E. T and V.
F. U and V.

- S is not a s.s. because if $f(x) = x+1$ $(\forall x \in \mathbb{R})$, $g(x) = x-1$ $(\forall x \in \mathbb{R})$, then $f, g \in S$, but $(f+g)(x) = 2x$ $(\forall x \in \mathbb{R})$ does not belong to S $((f+g)(-1)\cdot(f+g)(1) = -4 \neq 0)$.

- U is not a s.s. of $F(\mathbb{R})$ since it does not contain the zero function.

(By elimination, then, the correct answer must be E. To see directly that T and V are indeed subspaces, simply run the subspace test on each.

3. Suppose $\{u, v, w\}$ is a set of vectors in a vector space V. Which of the following statements is equivalent to

"$\{u, v, w\}$ is linearly independent." (*)

I. None of the vectors u, v or w is a multiple of any other single vector in $\{u, v, w\}$. ✗

II. None of the vectors u, v or w is a linear combination of the other vectors in $\{u, v, w\}$. ✓

III. If $a = b = c = 0$, then $au + bv + cw = 0$. This is a true statement, but is not equivalent to (*)

IV. If a, b, c are scalars, then $au + bv + cw = 0$ implies $a = b = c = 0$. ✓ (This is the definition)

V. Both $\{u, v\}$ and $\{v, w\}$ are linearly independent. ✗

A. I & II
B. I & III
C. II & III
D. II & V
E. II & IV
F. III & IV

(This question was on Test 1)

I is not equivalent to (*) since $\{(1,0), (0,1), (1,1)\}$ satisfies I but is not l.i.

III is not equivalent to (*) since $\{(0,0), (1,0), (2,0)\}$ satisfies III but is not l.i.

V The set $\{(1,0), (0,1), (1,1)\}$ satisfies V but is not l.i.

4. The dimension of $\{A \in \mathbf{M}_{22}(\mathbf{R}) \mid A = A^t\}$ is:

A. 0
B. 1
C. 2
D. 3
E. 4
F. 5

$$W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{22} \;\middle|\; \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \right\}$$

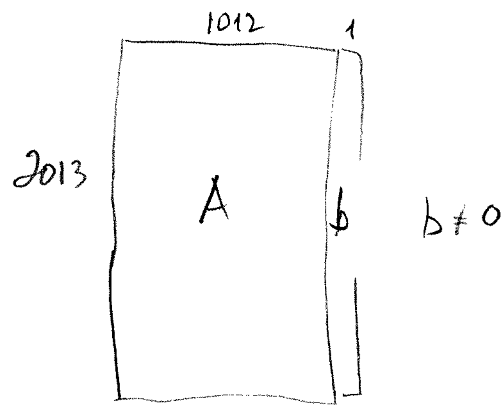
$$= \left\{ \begin{bmatrix} a & b \\ b & d \end{bmatrix} \in M_{22} \;\middle|\; a, b, d \in \mathbb{R} \right\}$$

$$= \operatorname{span}\left\{ \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{M_1}, \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{M_2}, \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{M_3} \right\}.$$

Moreover, $\{M_1, M_2, M_3\}$ is l.i., so $\dim W = 3$.

5. For a non-homogeneous system of 2013 equations in 1012 variables, answer the following three questions:

I ◦ Can the system be inconsistent?
II ◦ Can the system have infinitely many solutions?
III ◦ Can the system have exactly one solution?



A. No, Yes, No.
B. Yes, Yes, Yes.
C. Yes, Yes, No.
D. No, No, No.
E. Yes, No, Yes.
F. No, No, Yes.

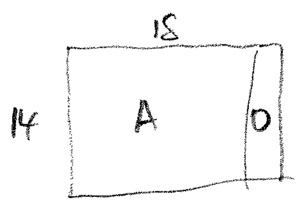
(I) YES: e.g. $[A|b] = \left[\begin{array}{c|c} 0 & \begin{matrix}1\\0\\\vdots\\0\end{matrix}\end{array}\right]$. (II) Yes: eg $[A|b] = \left[\begin{array}{ccc|c} 1 & 0 \cdots & 0 & 1 \\ 0 & \cdots & 0 & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & 0 & 0 \end{array}\right]$

has infly many sol'ns with 1011 parameters.

(III) Yes, Since rank $A = 1012 = \#$ variables is possible eg $[A|b] = \left[\begin{array}{c|c} I_{1012} & \begin{matrix}1\\0\\\vdots\end{matrix} \\ 0 & 0 \end{array}\right]$

6. If the coefficient matrix A in a homogeneous system of 14 equations in 18 variables is known to have rank 8, how many parameters are there in the general solution?

A. 4
B. 6
C. 8
D. 10
E. 18
F. 0



If rank $A = 8$, there are $18 - 8 = 10$ parameters in the general sol'n.

7. Let A be a fixed 4×4 matrix and let $B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$(r_i = i^{\text{th}} \text{ row of } A)$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} r_1 \\ r_2 + 2r_4 \\ r_3 \\ r_4 \end{bmatrix}}_{BA}$$

Then, BA can always be obtained from A by :

A. Adding twice the second row of A to the fourth row of A
B. Adding twice the fourth row of A to the second row of A
C. Adding the second row of A to the fourth row of A
D. Adding the fourth row of A to the second row of A
E. Adding twice the second column of A to the fourth column of A
F. Adding twice the fourth column of A to the second column of A

8. If $M = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix}$, which one of the following statements is true ?

A. M is not invertible.

B. The 3^{rd} row vector of M^{-1} is $(1, -1, 1)$.

C. The 1^{st} row vector of M^{-1} is $(1, 2, 1)$.

D. The 2^{nd} row vector of M^{-1} is $(1, 0, -1)$.

E. The 2^{nd} column vector of M^{-1} is $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$.

F. M^{-1} is invertible, but none of B, C, D, or E is true.

$$[M \mid I_3] = \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right]$$

Further row operations to bring the LHS to I_3 do not change the 3^{rd} row, so we know A^{-1} exists and its 3^{rd} row vector is $(1,-1,1)$.

9. Let A be an $n \times n$ matrix. Among the following statements, one is <u>not</u> equivalent to the other five. Which one is it?

A. A is invertible.

B. For any vector $b \in \mathbf{R}^n$, the system $Ax = b$ has a unique solution $x \in \mathbf{R}^n$.

C. The rows of A are linearly dependent.

D. A can be row-reduced to the identity matrix I_n.

E. The rank of A is n.

F. The columns of A span $\mathbf{R}^n$.

The rows of an invertible matrix are linearly <u>in</u>dependent.

$2C_1 + C_3 \to C_3$

10. If $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 3$, find $\begin{vmatrix} a & 4g & d-2a \\ b & 4h & e-2b \\ c & 4i & f-2c \end{vmatrix}$. $= \begin{vmatrix} a & 4g & d \\ b & 4h & e \\ c & 4i & f \end{vmatrix}$

$= 4 \begin{vmatrix} a & g & d \\ b & h & e \\ c & i & f \end{vmatrix} = -4 \begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix}$

$= -4 \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -4(3) = -12$

A. 24

B. -24

C. 12

D. -12

E. 6

F. -6

11. a) Let $k \in \mathbf{R}$ and consider the linear system in the unknowns x, y and z:

$$\begin{array}{rcrcrcl} x & + & 2y & + & z & = & 0 \\ x & + & y & + & 2z & = & 0 \\ 2x & + & 3y & + & kz & = & 0 \end{array}$$

(Note: as this is a homogeneous system, it is always consistent.)

Find all values of k for which this system has

(i) a unique solution,
(ii) infinitely many solutions, and
(iii) no solutions.

We compute:

$$[A|0] = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & 1 & 2 & 0 \\ 2 & 3 & k & 0 \end{array}\right] \sim \cdots \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & k-3 & 0 \end{array}\right].$$

Thus (i) There is a unique sol^n $\Leftrightarrow$ rank $A = 3 \Leftrightarrow k \neq 3$

(ii) There are infinitely many sol^{ns} $\Leftrightarrow$ rank $A < 3 \Leftrightarrow k = 3$

(iii) This, as mentioned above, is a homogeneous system and always has $(0,0,0)$ as a sol^n. Thus, there are no values of k for which the system has no solutions

$\frac{1}{2}$ - correct RRE form for A

1 (i) (consistent with their RRE form)

1 (ii) (consistent with their RRE form)

$\frac{1}{2}$ (iii)

11. b) Find all solutions of

$$\begin{array}{rcrcrcl} 2x & + & y & + & z & = & 14 \\ 3x & + & 3y & & & = & 15 \\ 5x & + & 4y & + & z & = & 29 \end{array}$$

such that x, y **and** z **are integers with** $x \geq 3, y \geq 0$ **and** $z \geq 2$.

b) We compute: $\left[\begin{array}{ccc|c} 2 & 1 & 1 & 14 \\ 3 & 3 & 0 & 15 \\ 5 & 4 & 1 & 29 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 5 \\ 2 & 1 & 1 & 14 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 5 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \begin{array}{c} \\ \left[\begin{array}{ccc|c} 1 & 0 & 1 & 9 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$. Thus the

s (above the third column)

general solution (ignoring constraints) is

$\{(9-s, -4+s, s) \mid s \in \mathbb{R}\}$, ie. $x = 9-s$, $y = -4+s$, $z = s$. ; $s \in \mathbb{R}$.

Implementing the constraints, we see:

$x \geq 3 \iff 9-s \geq 3 \iff 6 \geq s$
$y \geq 0 \iff -4+s \geq 0 \iff s \geq 4$ $\Big\} *$
$z \geq 2 \iff s \geq 2$

To satisfy $*$, we must have $6 \geq s \geq 4$. Since s (and hence $9-s$, $-4+s$) must be an integer, the only possible values for s are 4, 5 and 6. Hence the possible solutions satisfying the constraints are

$(5, 0, 4)$, $(4, 1, 5)$ and $(3, 2, 6)$. 2

1/2 - Correct RRE form 1/2 gen'l sol'n consistent with their RRE form

12. $W = \text{span}\{\overset{v_1}{(1,0,1,0)}, \overset{v_2}{(0,1,0,1)}, \overset{v_3}{(0,0,0,1)}, (1,1,1,0)\} \subset \mathbf{R}^4. = \{(x,y,z,w) \mid x-z=0\}$

$1\frac{1}{2}$ a) Find a basis of W which is a subset of the given spanning set.

2 b) Find an orthogonal basis of W.

$1\frac{1}{2}$ c) Find the best approximation to $(0,1,-1,1)$ in W.

1 d) Extend your basis of W in (b) to a basis of $\mathbf{R}^4$.

a) We use the column space algorithm: $\begin{bmatrix} 1&0&0&1\\ 0&1&0&1\\ 1&0&0&1\\ 0&1&1&0 \end{bmatrix} \sim \begin{bmatrix} 1&0&0&1\\ 0&1&0&1\\ 0&0&1&-1\\ 0&0&0&0 \end{bmatrix}$.

Hence $\{v_1, v_2, v_3\}$ is a basis for W which is a subset of the given spanning set. (1) - any correct basis ($\frac{1}{2}$) - just^n

b) We apply Gram-Schmidt to v_1, v_2, v_3:

Set $u_1 = v_1 = (1,0,1,0)$

$$u_2 = v_2 - \frac{v_2 \cdot u_1}{\|u_1\|^2} u_1 = v_2 = (0,1,0,1)$$

(1/2) - Trying G.S. correctly

$$\tilde{u}_3 = v_3 - \cancel{\left(\frac{v_3 \cdot u_1}{\|u_1\|^2}\right)}^{0} u_1 - \frac{v_3 \cdot u_2}{\|u_2\|^2} u_2$$

(1) any orthogonal set of 3 vectors

($\frac{1}{2}$) any orthog set of 3 vectors in W.

$$= (0,0,0,1) - \frac{1}{2}(0,1,0,1)$$

$$= \left(0, -\tfrac{1}{2}, 0, \tfrac{1}{2}\right).$$

Set $u_3 = (0,-1,0,1)$. Then $\{u_1, u_2, u_3\}$ is an orthogonal basis of W.

c) We compute: $\text{proj}_W \underbrace{(0,1,-1,1)}_{v} = \left(\frac{v \cdot u_1}{\|u_1\|^2}\right) u_1 + \left(\frac{v \cdot u_2}{\|u_2\|^2}\right) u_2 + \left(\frac{v \cdot u_3}{\|u_3\|^2}\right) u_3$

(1) - Using orthog. proj^n correctly

$$= -\tfrac{1}{2} u_1 + \tfrac{2}{2} \cdot u_2 + 0 u_3$$

($\frac{1}{2}$) - correct answer

$$= \left(-\tfrac{1}{2}, 1, -\tfrac{1}{2}, 1\right) \text{ is the best approx.}$$

12d) We seek a vector u_4 s.t

$$4 = \operatorname{rank}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ & u_4 & & \end{bmatrix}$$

$$= \operatorname{rank}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 2 \\ & u_4 & & \end{bmatrix}.$$

Clearly, $u_4 = (0,0,1,0)$ satisfies $\operatorname{rank}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = 4$.

Thus, $\{u_1, u_2, u_3, (0,0,1,0)\}$ is a basis of $\mathbb{R}^4$ extending our basis in (b).

$\frac{1}{2}$ – any correct extension

$\frac{1}{2}$ – justn

13. $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$.

1/2 a) Show that the eigenvalues of A are 5 and -1.

1/2 b) Find a basis of $E_5 = \{v \in \mathbf{R}^3 \mid Av = 5v\}$.

2 c) Find a basis of $E_{-1} = \{v \in \mathbf{R}^3 \mid Av = -v\}$.

2 d) Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$. Explain why your choice of P is invertible.

a) $\operatorname{Det}(A-\lambda I) = \begin{vmatrix} 1-\lambda & 2 & 2 \\ 2 & 1-\lambda & 2 \\ 2 & 2 & 1-\lambda \end{vmatrix} \overset{\substack{-R_1+R_2\to R_2 \\ -R_1+R_3\to R_3}}{=} \begin{vmatrix} 1-\lambda & 2 & 2 \\ 1+\lambda & -(1+\lambda) & 0 \\ 1+\lambda & 0 & -(1+\lambda) \end{vmatrix} = (1+\lambda)^2 \begin{vmatrix} 1-\lambda & 2 & 2 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{vmatrix}$

$C_3+C_1\to C_1$

$= (1+\lambda)^2 \begin{vmatrix} 3-\lambda & 2 & 2 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix} = -(1+\lambda)^2 \begin{vmatrix} 3-\lambda & 2 \\ 1 & -1 \end{vmatrix} = -(1+\lambda)^2(\lambda-3-2) = -(1+\lambda)^2(\lambda-5)$.

Hence $\operatorname{Det}(A-\lambda I)=0 \Leftrightarrow \lambda=-1$ or $\lambda=5$. Thus the evals of A are -1 & 5.

b) We compute: $E_5 = \ker(A-5I) = \ker \begin{bmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{bmatrix} = \ker \begin{bmatrix} 1 & 1 & -2 \\ 0 & -6 & 6 \\ 0 & 6 & -6 \end{bmatrix}$ (1/2) consistent w RRE form

$= \ker \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} = \ker \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} = \{(s,s,s) \mid s \in \mathbb{R}\}$. (1/2) Hence a

basis for E_5 is $\{(1,1,1)\}$. consistent w gen'l sol^n (The vector $(1,1,1)$ being the only basic sol^n)

c) We compute: $E_{-1} = \ker(A+I) = \ker \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} = \ker \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \{(-s-t, s, t) \mid s,t \in \mathbb{R}\}$

(1/2) correct RRE form (1/2) gen'l sol^n consistent w/ RRE form (1) - basis consistent w gen'l sol^n

Hence a basis is $\{(-1,1,0), (-1,0,1)\}$ (the set of the 2 basic solutions.)

d) Set $P = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. We know that $AP = PD$. Moreover,

$P \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & -2 \end{bmatrix}$ has rank 3, so P is invertible. Thus

$P^{-1}AP = D$.

(1) - P^* consistent w b and (c) (as long as it's 3x3)

(1/2) - D consistent w P

(1/2) - just^n that P is invertible

(* - (1/2) if P^t given.)

14. Let $u = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ and define a linear transformation $S : \mathbf{R}^3 \to \mathbf{R}^3$ by

$$S(v) = (u \cdot v)\, u, \qquad v \in \mathbf{R}^3,$$

where "$u \cdot v$" denotes dot product of u and v. *(You do not have to prove that S is linear.)*

$\frac{1}{2}$ a) If $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbf{R}^3$, show that $S(v) = \begin{bmatrix} x-y+z \\ -x+y-z \\ x-y+z \end{bmatrix}$.

$\frac{1}{2}$ b) Find a 3×3 matrix A such that $S(v) = Av$, where Av denotes the matrix product of A and v.

$2\frac{1}{2}$ c) Find a basis for $\ker S = \{v \mid S(v) = 0\}$, and give a complete geometric description of $\ker S$.

$2\frac{1}{2}$ d) Find a basis for $\operatorname{im} S = \{S(v) \mid v \in \mathbf{R}^3\}$, and give a complete geometric description of $\operatorname{im} S$.

a) Since $u \cdot v = x - y + z$, $S(v) = (x-y+z)\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} x-y+z \\ -x+y-z \\ x-y+z \end{bmatrix}$.

$\frac{1}{2}$

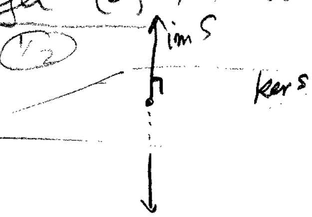
b) We know that $A = [Se_1 \ Se_2 \ Se_3]$ (block column form) $= \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$. $\frac{1}{2}$

c) We know that $\ker S = \ker A = \ker \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} = \ker \begin{array}{c} \begin{array}{ccc} & s & t \end{array} \\ \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$ $(*)$

$\frac{1}{2}$ basis consistent w (b)
$\frac{1}{2}$ justn.

$= \{(s-t, s, t) \mid s, t \in \mathbb{R}\}$, so $\{(1,1,0), (-1,0,1)\}$ is a basis of $\ker S$. Since $\ker S = \{(x,y,z) \mid x - y + z = 0\}$ (from $(*)$, or directly from the defn), $\ker S$ is the plane through $(0,0,0)$ with normal $(1,-1,1)$.
$\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$

d) We know that $\operatorname{im} S = \operatorname{col} A = \operatorname{span}\{(1,-1,1)\}$, by calculations in (c) (and the column space algorithm). Hence $\{(1,-1,1)\}$ is a basis of $\operatorname{im} S$, which is a line through $(0,0,0)$ with direction $(1,-1,1)$.
$\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$



($\Leftarrow$ in this case.)

$\frac{1}{2}$ basis consistent w b
$\frac{1}{2}$ justn

$4 \times 1.5 \quad = 4 \times (\tfrac{1}{2}\text{-correct} + 1\text{-justn})$

15. State whether each of the following is (always) true, or is (possibly) false, in the box after the statement.

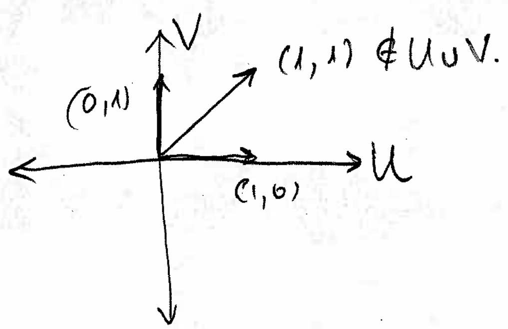
- If you say the statement may be false, you **must give an explicit example - with numbers!**
- If you say the statement is true, you must give a clear explanation - by quoting a theorem presented in class, or by giving a *proof valid for every case.*

a) If U and V are subspaces of $\mathbf{R}^2$, then their union

$$U \cup V = \{w \in \mathbf{R}^3 \mid w \in U \text{ or } w \in V\}$$

is also a subspace of $\mathbf{R}^2$.

Let $U = \{(x,0) \mid x \in \mathbb{R}\} = \text{span}\{(1,0)\}$ and $V = \{(0,y) \mid y \in \mathbb{R}\} = \text{span}\{(0,1)\}$ be the x and y-axes respectively.



Then $(1,0) \in U$ and so $(1,0) \in U \cup V$, and $(0,1) \in V$ and so $(0,1) \in U \cup V$, but $(1,0) + (0,1) = (1,1) \notin U \cup V$ (which is pictured.)

ANSWER $\boxed{\text{FALSE}}$

b) $\begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$ is diagonalizable.

This 2×2 matrix is diagonalizable because it has 2 distinct eigenvalues, namely, 2 and 3. (These are its evals because these are the diagonal entries of this lower triangular matrix).

ANSWER $\boxed{\text{TRUE.}}$

15 (cont.)

c) If 0 is an eigenvalue of 7×7 matrix A, then A is invertible.

Indeed, this statement is <u>always</u> false! If 0 is an eval. of A, then $0 = \det(A - 0I) = \det A$, so A cannot be invertible. (An example is $A = 0_{7\times 7}$!)

ANSWER $\boxed{\text{FALSE}}$

d) If A is an $m \times n$ matrix with $n > 1$ and $m > 1$, and if the reduced row echelon form of A has a row of zeros, then rank $A < n$.

Let $A = \begin{array}{cc} & n \\ m & \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \end{array}$. Then A is in reduced row echelon form, has a row of zeros (row 3) but rank $A = 2 = n$.

ANSWER $\boxed{\text{FALSE}}$

16. (4 bonus marks) Make sure you finish and check the rest of the paper before trying this. Bonus marks are much harder to earn.

Prove carefully that if A is an 11×11 anti-symmetric matrix (i.e. $A^t = -A$) , then 0 is an eigenvalue of A. (Your proof must be valid for all possible 11×11 anti-symmetric matrices.)

We show that $0 = \det(A - 0I) = \det A$, as follows:

Since $\det A = \det A^t$ (for all matrices) and here

$\det(-A) = (-1)^{11} \det A = -\det A$, $A^t = -A$ implies

$\det A = \det(A^t) = \det(-A) = -\det A$. Hence $\det A = 0$.

Thus, $\det(A - 0I_{11}) = 0$, & 0 is an eigenvalue of A.

(1) Knowing "0 is an eval $\Leftrightarrow$ A is not invertible"

+ (1) - "det A = 0" ← some attempt + (1) correct

+ (1) - well written