MA122 Mock Final Exam

Name: _____

Time Allowed: 150 minutes
Total Value: 100 marks
Number of Pages: 9

Instructions:

Cheat Sheet: One 8.5" × 11" page of study notes (both sides) is allowed as a reference while completing the mock test. Please note, that the cheat sheet is permitted for the mock test only!!

Check that your test paper has no missing, blank, or illegible pages. Note that test questions appear on ***both*** *sides of the paper.*

Answer in the spaces provided.

Show all your work. Insufficient justification will result in a loss of marks.

1. [5 *marks*] Consider the matrix

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & -4 \\ -2 & 0 & 2 \end{bmatrix}$$

Determine A^{-1}.

2. [6 *marks*] Determine which value(s) of a will allow the linear system $A\mathbf{x} = \mathbf{0}$ with

$$A = \begin{bmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ -6 & 7a & -7 & 4 \\ -9 & 8 & 0 & 9 \end{bmatrix}$$

to have nontrivial solutions.

3. [4 *marks*] If M is an $n \times n$ matrix, show that $\det\left(MM^T\right) \geq 0$.

4. [7 *marks*] Suppose that the vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3 \in \mathbb{R}^4$ are non-zero and mutually orthogonal (i.e., they are all perpendicular to one another). Prove that they must be linearly independent.

   [Hint: Consider $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = \mathbf{0}$ and take the dot product with $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ in turn.]

5. [4 *marks*] Prove the identity for any vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$: $\|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2 = 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2$.

6. [8 *marks*] Let $A\,(6,-1,3)$, $B\,(0,0,5)$, $C\,(-2,3,2)$ and $D\,(4,2,0)$ be the vertices of parallelogram $ABCD$.

   (a) Determine the cosine of the angle at vertex A.

   (b) Determine the area of the parallelogram.

7. [8 *marks*] Find the parametric equations of the line l which passes through the point $P\,(2,1,0)$ and is perpendicular to the plane $\pi : 3x + 4y - z = -3$. Determine the point at which the line intersects the plane.

8. [8 *marks*]

(a) Express, in general form, an equation of the plane π that passes through the points $P(2,1,-3)$ and $Q(3,2,-2)$, and is perpendicular to the plane $\pi_0 : 2x - 4y - z = -1$.

(b) Determine whether the plane π_0 given in part(a) is parallel or perpendicular to the line (or neither!).

$$l: \quad x = -1 + 4t, \quad y = 3 + 4t, \quad z = -2 - 8t, \quad t \in \mathbb{R}.$$

9. [5 *marks*] Determine if the set $\left\{1 - x^2, 3x, 1 + 4x - x^2\right\}$ spans $\mathbf{P}_2$ (all polynomials of degree 2).

10. [10 *marks*]

(a) Verify that the following set of vectors in $\mathbb{R}^4$ are linearly independent:

$$\{\ (3,0,-3,6)\,,\ \ (0,2,3,1)\,,\ \ (0,-2,-2,0)\,,\ \ (-2,1,2,1)\ \}$$

(b) Express the vector $(1,2,3,4)$ as a linear combination of the vectors contained in the set given in part(a).

11. [5 *marks*] Let T be a linear transformation with the following matrix representation:

$$[T] = \begin{bmatrix} 2 & -4 & 3 & -1 \\ 1 & -2 & 1 & 0 \\ 0 & -6 & 1 & 4 \end{bmatrix}$$

(a) State the domain and codomain of T.

Domain: _____ Codomain: _____

(b) Determine each of the following, if possible:

(i) $T(1, 2, 3, 4)$ (ii) $T(-1, 0, 1)$

12. [8 *marks*] Consider the two linear operators, $T : \mathbb{R}^3 \to \mathbb{R}^3$ and $S : \mathbb{R}^3 \to \mathbb{R}^3$ defined by:

T : orthogonal projection on the yz-plane,
$S(x_1,\ x_2,\ x_3) = (x_3,\ x_1 + x_2 + x_3,\ -x_2)$.

(a) State the matrix respresentation for the composition $S \circ T$.

(b) Prove that S is one-to-one.

(c) Find $S^{-1}(w_1,\ w_2,\ w_3)$.

13. [7 *marks*] Let $V = \mathbb{R}^2$. Define scalar multiplication in the usual way, but define vector addition as follows:

$$(u_1, u_2) + (v_1, v_2) = (u_1 v_1, u_2 v_2)$$

for any $\mathbf{u}, \mathbf{v} \in V$.

(a) Determine the zero vector (additive identity), fully justifying your answer.

(b) Determine, if possible, the negative (additive inverse) for the vector $(3, 4)$ and for the vector $(1, 0)$.

(c) Is V a vector space under this definition for scalar multiplication and addition? Justify your answer.

14. [6 *marks*] Determine whether the following set A is a subspace of $\mathbf{M}_{2\text{x}2}$ (all 2×2 matrices), under the usual rules for addition and scalar multiplication:

$$A = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \;\middle|\; a, b, c \in \mathbb{R} \right\}.$$

15. [10 *marks*] Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$ where

$$A = \begin{bmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$