# Econ 6311 Midterm
(October 22, 2020)

1. Let $Y = m(X) + u$, in which $m(X)$ is the conditional expectation function (CEF) of $Y$ in terms of $X$ (i.e., $m(X) = E(Y|X)$). Suppose $m(X)$ is linear in $X$ such that $m(X) = X'\gamma$.

   (a) Show that $E(Xu) = 0$.

   - It is easy to show that $E(u|X) = 0$. Thus, we have

   $$\begin{aligned} E(Xu) &= E(E\{Xu|X\}) \\ &= E(XE\{u|X\}) = 0. \end{aligned}$$

   (b) Does the OLS estimator, obtained from regressing $Y$ on $X$, converge to $\gamma$? If yes, prove it. If no, explain the reason.

   - The probability limit of the OLS is

   $$\gamma^* = E(XX')^{-1} EXY.$$

   Then,

   $$\begin{aligned} X'\gamma^* &= X'E(XX')^{-1} EXY \\ &= X'E(XX')^{-1} E(X[X'\gamma + u]) \\ &= X'\gamma + X'E(XX')^{-1} \underbrace{E(Xu)}_{=0}, \end{aligned}$$

   which implies $\gamma = \gamma^*$.

   (c) Consider a causal model $Y = X\beta + \varepsilon$, the right hand side of which causes $Y$. Thus, $\beta$ represents the change of $Y$ by one unit increase in $X$, provided other factors are held constant. What is the relationship between $\beta$ and $\gamma$? Are they the same in general? Otherwise, what condition do we need to have $\beta = \gamma$?

   - Since

   $$\begin{aligned} \gamma &= \gamma^* = E(XX')^{-1} EXY \\ &= E(XX')^{-1} EX(X\beta + \varepsilon) \\ &= \beta + E(XX')^{-1} EX\varepsilon, \end{aligned}$$

   we need $EX\varepsilon = 0$ to have $\beta = \gamma$.

2. Suppose we are interested in estimating the effect of student-teacher ratio (STR) on students' test scores.

$$TestScore = \beta_0 + \beta_1 STR + u.$$

We have the average test scores of the 5th grade in all schools in CT, but the STRs of the 5th grade are not available. Instead, we have the STRs of the 2nd grade. Let

$$STR = STR_{(2)} + \eta$$

where $STR$ denotes the STR of the 5th grade and $STR_{(2)}$ denotes the STR of the 2nd grade. We assume

$$E(u) = E(\eta) = 0$$

and

$$E(STR \cdot u) = E(STR \cdot \eta) = E(u\eta) = 0.$$

(a) Suppose we run the OLS based on the following regression

$$TestScore = \beta_0 + \beta_1 STR_{(2)} + v \tag{1}$$

where $TestScore$ denotes the average test score of the 5th grade. Is the OLS estimator for $\beta_1$, consistent? If not, what is the direction of bias?

- It is not consistent - measurement error bias. Let $X^* = STR$, $X = STR_{(2)}$ and $Y = TestScore$. Then,

$$\begin{aligned} Y &= \beta_0 + \beta_1 X + v \\ &= \beta_0 + \beta_1 (X^* + \eta) + v = \beta_0 + \beta_1 X^* + \underbrace{\beta_1 \eta + v}_{=u}. \end{aligned}$$

Thus, $v = u - \beta_1 \eta$. Since

$$EXv = E(X^* + \eta)(u - \beta_1 \eta) = -\beta_1 \sigma_\eta^2 \neq 0,$$

if $\beta_1 > 0$, $EXv < 0 \to$ the bias is negative, and if $\beta_1 < 0$, $EXv > 0 \to$ the bias is positive.

(b) Suppose the STR of the 6th grade, denoted by $STR_{(6)}$, is also available, and we assume

$$STR_{(6)} = STR + e$$

where $E(e) = 0$ and $E(STR \cdot e) = E(ue) = E(e\eta) = 0$. Consider an two stage least squares estimator (or IV estimator) based on (1) that uses $STR_{(6)}$ as an IV. Discuss if this estimator is consistent.

- Let $Z = STR_{(6)}$. We can show that $Z$ is relevant and valid (exogenous), which implies that this estimator is consistent. First,

$$\begin{aligned} Cov(Z, X) &= Cov(X^* + e, X^* + \eta) \\ &= Var(X^*) \neq 0, \end{aligned}$$

so $Z$ is relevant. Second

$$Cov(Z_i, v) = Cov(X^* + e, v) = Cov(X^*, v) + Cov(e, v) = 0,$$

so $Z$ is valid.

3. Consider a linear regression model

$$Y_i = X_i'\beta + u_i \text{ and } E(u_i) = 0,$$

where $X_i$ is a $(k \times 1)$ vector and $E(X_i u_i) \neq 0$. Suppose we have a $(r \times 1)$ vector of instruments $Z_i$ and $r > k$. We also assume $E(u_i^2 | Z_i) = \sigma^2$.

(a) Describe the two stage least squares estimation procedure. Is this estimator efficient? Yes. Using the notation we used in class, we can write the TSLS estimator as

$$\hat{\beta}_{TSLS} = \left(\hat{\Sigma}_{XZ}\hat{\Sigma}_{ZZ}^{-1}\hat{\Sigma}_{ZX}\right)^{-1}\hat{\Sigma}_{XZ}\hat{\Sigma}_{ZZ}^{-1}\frac{1}{n}\sum_{i=1}^{n} Z_i Y_i.$$

- The efficient GMM estimator use the weight matrix

$$\begin{aligned}\hat{\Omega} &= \widehat{Var}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} Z_i u_i\right) \\ &= \hat{\sigma}^2\frac{1}{n}\sum_{i=1}^{n} Z_i Z_i'\end{aligned}$$

under the homoskedasticity assumption above. Then, the efficient GMM estimator is equivalent to $\hat{\beta}_{TSLS}$ since

$$\hat{\Omega} = \hat{\sigma}^2\hat{\Sigma}_{ZZ}.$$

(b) Describe the procedure of the overidentification test. Construct the test statistic, and explain the null and alternative hypothesis and how to obtain the critical values. Do not derive the asymptotics of this test as in the problem set.

- $H_0 : E(Z_i u_i) = 0$ and $H_1 : E(Z_i u_i) \neq 0$. The test statistic is

$$J = \left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} Z_i\hat{u}_i\right)'\left(\hat{\sigma}^2\frac{1}{n}\sum_{i=1}^{n} Z_i Z_i'\right)^{-1}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} Z_i\hat{u}_i\right)$$

where $\hat{u}_i = Y_i - X_i'\hat{\beta}_{TSLS}$ and $\hat{\sigma}^2 = n^{-1}\sum_{i=1}^{n}\hat{u}_i^2$. We reject $H_0$ at $100\alpha\%$ significance level if $J > \chi^2_{1-\alpha}(r-k)$ where $r$ is the number of IVs and $k$ is the dimension of $\beta$.

4. Suppose that we have a model

$$Y_{it} = X_{it}'\beta + \alpha_i + u_{it}$$

where $X_{it} =$ is a $(k \times 1)$ vector of regressors. We introduce the following set of standard FE assumptions.

1. $E(u_{it} | X_i, \alpha_i) = 0,$
2. $Rank\left(E\left(X_i - \bar{X}_i\right)'\left(X_i - \bar{X}_i\right)\right) = k$
3. $E(u_i u_i' | X_i, \alpha_i) = \sigma_u^2 I_T.$

(a) The "first differencing" method uses the following transformation

$$\begin{aligned} y_{it} &= X_{it}'\beta + \alpha_i + u_{it} \\ -\ y_{it-1} &= X_{it-1}'\beta + \alpha_i + u_{it-1} \end{aligned}$$

so that we have

$$\Delta y_{it} = \Delta X_{it}'\beta + \Delta u_{it}$$

The OLS estimator of this transformed model is called "first differencing estimator," denoted by $\hat{\beta}_{FD}$. Show that

$$\hat{\beta}_{FD} = \hat{\beta}_{FE}.$$

when $T = 2$.

- See the Practice Question 6.

(b) Suppose we are interested in $\{\alpha_i\}$ and estimate $\alpha_i$ with

$$\hat{\alpha}_i = \bar{Y}_{i\cdot} - \bar{X}_{i\cdot}'\hat{\beta}$$

where $\bar{Y}_{i\cdot} = T^{-1}\sum_{t=1}^{T} Y_{it}$ and $\bar{X}_{i\cdot} = T^{-1}\sum_{t=1}^{T} X_{it}$. Explain if $\hat{\alpha}_i$ is consistent under large $n$ asymptotics, i.e., as $n \to \infty$ with $T$ fixed?

- Since $\bar{Y}_{i\cdot} = \bar{X}_{i\cdot}'\beta + \alpha_i + \bar{u}_{i\cdot}$ and $\hat{\beta}_{FE} \to^P \beta$ as $n \to \infty$ with $T$ fixed, we have

$$\begin{aligned} \hat{\alpha}_i &= \bar{X}_{i\cdot}'\beta + \alpha_i + \bar{u}_{i\cdot} - \bar{X}_{i\cdot}'\hat{\beta}_{FE} \\ &= \bar{X}_{i\cdot}'\left(\beta - \hat{\beta}_{FE}\right) + \alpha_i + \bar{u}_{i\cdot} \to^P \alpha_i + \bar{u}_{i\cdot} \neq \alpha_i \end{aligned}$$

because $\bar{u}_{i\cdot} = T^{-1}\sum_{i=1}^{T} u_{it}$ does not converges in probability to zero as $n$ increases if $T$ is fixed.