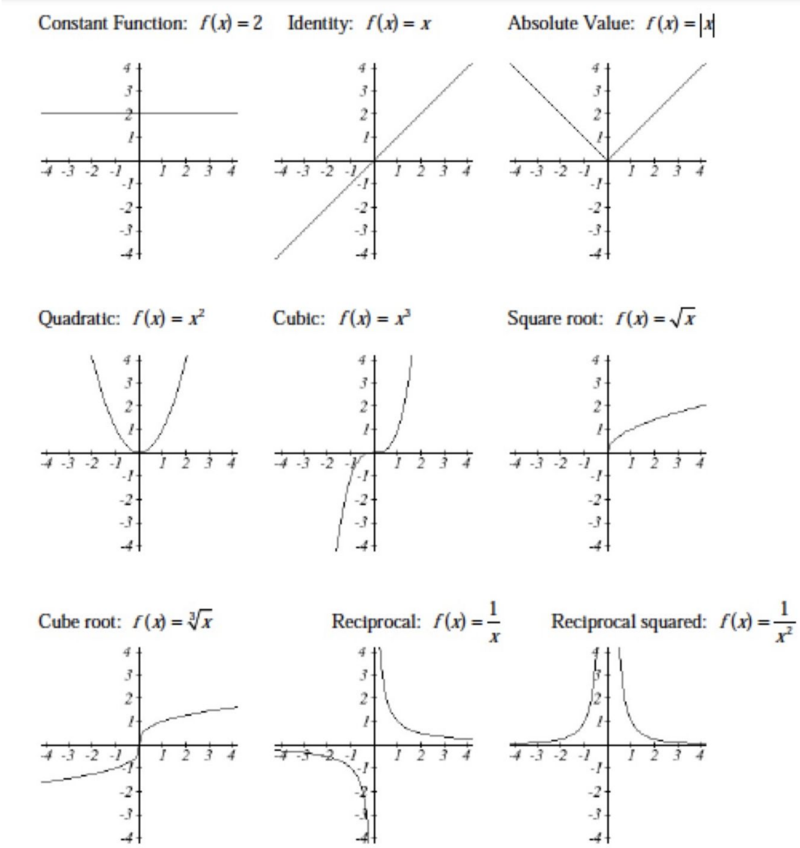
LECTURE 1

SECTION 1.1-1.2: PRE-CALCULUS REVIEW



Example 1.8. Solve for x: $|x^2 - 4| = 1$

$|a| = 1 \iff a = 1$ or $a = -1$

Answer:

$x^2 - 4 = 1$

add '4' for both sides: $x^2 = 1 + 4$

Simplify: $x^2 = 5$

$x = \pm\sqrt{5}$

or

$x^2 - 4 = -1$

add '4' for both sides: $x^2 = -1 + 4$

Simplify: $x^2 = 3$

$x = \pm\sqrt{3}$

$$\boxed{\therefore\ x = \pm\sqrt{3} \text{ or } \pm\sqrt{5}}$$

Example 1.9. Find the domain of each function

a) $f(x) = 2\sqrt{x+4}$

b) $g(x) = \frac{3}{6-3x}$

Answer:

a) Cannot take the square root of a negative number, so the inside of the square root to be non-negative

$$x + 4 \geq 0$$

$$x \geq -4$$

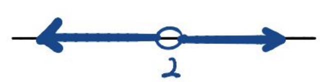
$\therefore$ the domain of $f(x) = 2\sqrt{x+4}$ is $[-4, +\infty)$

Actually: for $f(x) = x^{\frac{1}{n}}$, where n is an even number, its domain is $[0, +\infty)$

b) the denominator cannot be zero

$\Rightarrow 6 - 3x \neq 0 \quad \Rightarrow x \neq 2$

$\therefore$ the domain of $g(x) = \frac{3}{6-3x}$ is $(-\infty, 2) \cup (2, +\infty)$



Example 1.10. Find the composition $f \circ g$ and $g \circ f$, where $f(x) = \frac{x}{x+1}$ and $g(x) = x^2$

Answer

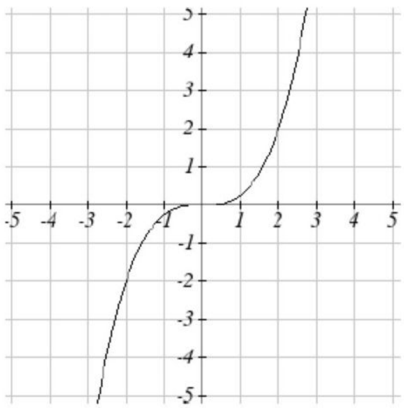
$$f \circ g = f(g(x)) = f(x^2) = \boxed{\frac{x^2}{x^2+1}}$$

$$g \circ f = g(f(x)) = g\left(\frac{x}{x+1}\right) = \boxed{\left(\frac{x}{x+1}\right)^2}$$

Hence $f \circ g \neq g \circ f$

	Vertical Stretch	Horizontal Stretch
$g(x)$	$kf(x)$	$f(kx)$
If $k > 1$	vertical stretch by a factor of k	horizontal shrink by a factor of $1/k$
If $0 < k < 1$	vertical shrink by a factor of k	horizontal stretch by a factor of $1/k$
If $k < 0$	combination of a vertical stretch or shrink with a vertical reflection	combination of a horizontal stretch or shrink with a horizontal reflection

Example 1.13. The graph below is a vertical stretch/shrink of the toolkit function $f(x) = x^3$. Relate this new function $g(x)$ to $f(x)$, then find a formula for $g(x)$.



Answer: Find $g(x) = kf(x)$

To determine the formula for $g(x)$. it is helpful to look for a point on the graph

In the graph. It is clear there is a point $(2,2)$

so $g(2) = 2$

$g(2) = kf(2) = k \cdot 2^3 = 8k$

Using the fact on the graph $g(2) = 2$

$8k = 2$

$k = \frac{2}{8} = \frac{1}{4}$

$$\boxed{\therefore\ g(x) = \frac{1}{4} f(x) = \frac{1}{4} x^3}$$

Intersections of Lines:

Example 1.14. The supply, in thousands of items, for custom phone cases can be modeled by the equation $s(p) = 1 + 2p$, while the demand can be modeled by $d(p) = 9 - 3p$, where p is in dollars. Find the equilibrium price and quantity, the intersection of the supply and demand curves.

Answer:

setting $s(p) = d(q)$

$$1 + 2p = 9 - 3p$$

$$5p = 8$$

$$p = 1.6$$

The equilibrium price is 1.6

Substitute $p = 1.6$ into $s(p)$

$$s(1.6) = 1 + 2 \cdot 1.6 = 4.2$$

or Substitute $p = 1.6$ into $d(p)$

$$d(1.6) = 9 - 3 \cdot 1.6 = 4.2$$

$\therefore$ the intersection of the supply and demand curves is $(1.6, 4.2)$

Example 1.15. Find intersections of the graphs of $f = 2 - x^2)$ and $g(x) = 2x^2 + 2x - 3$

Answer:

setting $f(x) = g(x)$

$$2 - x^2 = 2x^2 + 2x - 3$$

$$3x^2 + 2x - 5 = 0$$

By quadratic formula: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

where $a = 3$, $b = 2$ and $c = -5$

$$x = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 3 \cdot (-5)}}{2 \cdot 3} = 1 \text{ or } \left(-\frac{5}{3}\right)$$

Substitute $x = 1$ and $-\frac{5}{3}$ into $f(x)$

$$f(1) = 2 - 1^2 = 1$$

$$f\left(-\frac{5}{3}\right) = 2 - \left(-\frac{5}{3}\right)^2 = -\frac{7}{9}$$

or Substitute $x = 1$ and $-\frac{5}{3}$ into $g(x)$

$$g(1) = 2 \cdot 1^2 + 2 \cdot 1 - 3 = 1$$

$$g\left(-\frac{5}{3}\right) = 2\left(-\frac{5}{3}\right) + 2\left(-\frac{5}{3}\right) - 3 = -\frac{7}{9}$$

$\therefore$ two intersection points are $(1, 1)$ and $\left(-\frac{5}{3}, -\frac{7}{9}\right)$

Composition of Functions

When the output of one function is used as the input of another, we call the entire operation a **composition of functions**. We write *f(g(x))*, and read this as "*f* of *g* of *x*" or "*f* composed with *g* at *x*".

An alternate notation for composition uses the composition operator: $\circ$

$(f \circ g)(x)$ is read "*f* of *g* of *x*" or "*f* composed with *g* at *x*", just like *f(g(x))*.

Example 3

Given $f(t) = t^2 - t$ and $h(x) = 3x + 2$, evaluate $f(h(1))$.

Since the inside evaluation is $h(1)$ we start by evaluating the *h(x)* function at 1:

$h(1) = 3(1) + 2 = 5$

Then $f(h(1)) = f(5)$, so we evaluate the *f(t)* function at an input of 5:

$f(h(1)) = f(5) = 5^2 - 5 = 20$

LECTURE 2

PRE-CALCULUS REVIEW II

EXPONENTIAL FUNCTIONS

Point-Slope Equation of a Line: An equation for the line passing through the point (x_1, y_1) with slope m can be written as

$$\boxed{y - y_1 = m(x - x_1)}$$

This is called the **point-slope form of a line**. It is a little easier to write if you know a point and a slope, but requires a bit of work to rewrite into a slope-intercept form.

- a point : (x_1, y_1)
- a slope : m

$$\boxed{y - y_1 = m(x - x_1)} \quad \rightarrow \quad \boxed{y = \underset{\text{slope}}{m}x + \overbrace{(y_1 - mx_1)}^{\text{Intercept}}}$$

The point-slope form

The slope-Intercept form

Slope and Increasing/Decreasing

m is the constant rate of change of the function (also called **slope**). The slope determines if the function is an increasing function or a decreasing function.

$f(x) = b + mx$ is an **increasing** function if $m > 0$

$f(x) = b + mx$ is a **decreasing** function if $m < 0$

If $m = 0$, the rate of change zero, and the function $f(x) = b + 0x = b$ is just a horizontal line passing through the point (0, *b*), neither increasing nor decreasing.

Rules of Exponents or Laws of Exponents	
Multiplication Rule	$a^x \times a^y = a^{x+y}$
Division Rule	$a^x \div a^y = a^{x-y}$
Power of a Power Rule	$\left(a^x\right)^y = a^{xy}$
Power of a Product Rule	$(ab)^x = a^x b^x$
Power of a Fraction Rule	$\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$
Zero Exponent	$a^0 = 1$
Negative Exponent	$a^{-x} = \frac{1}{a^x}$
Fractional Exponent	$a^{\frac{x}{y}} = \sqrt[y]{a^x}$

Example 2.7. A ball is thrown upwards from the top of a 40-foot high building at a speed of 80 feet per second. The balls height above ground can be modeled by the equation $H(t) = -16t^2 + 80t + 40$. When does the ball hit the ground?

Answer:

when the ball hits the ground, $H(t) = 0$

solve $H(t) = 0$

$-16t^2 + 80t + 40 = 0$

$a = -16$, $b = 80$, $c = 40$

$$t = \frac{-80 \pm \sqrt{80^2 - 4 \cdot (-16)(40)}}{2 \cdot (-16)}$$

$= 5.458$ or $\cancel{-0.458}$ (no applicable)

$\boxed{\therefore \text{ when } t = 5.458\text{s, the ball hit the ground.}}$

Horizontal Asymptote of Rational Functions:

The **horizontal asymptote** of a rational function can be determined by looking at the degrees of the numerator and denominator.

- Degree of denominator $>$ degree of numerator: Horizontal asymptote at $y = 0$
- Degree of denominator $<$ degree of numerator: No horizontal asymptote
- Degree of denominator $=$ degree of numerator: Horizontal asymptote at ratio of leading coefficients.

Example 2.11. A large mixing tank currently contains 100 gallons of water, into which 5 pounds of sugar have been mixed. A tap will open pouring 10 gallons per minute of water into the tank at the same time sugar is poured into the tank at a rate of 1 pound per minute. Find the concentration (pounds per gallon) of sugar in the tank after t minutes.

Answer :

- The amount of water in the tank is changing linearly with a slope $m=10$

$$\text{water} = \underbrace{100}_{\text{initial value}} + \underbrace{10t}_{\text{slope}}$$

- The amount of sugar in the tank is also changing linearly with a slope $m=1$

$$\text{sugar} = 5 + t$$

$\therefore$ the concentration, C, will be the ratio of pounds of sugar to gallons of water

$$\boxed{C(t) = \frac{5+t}{100+10t}}$$

Exponential Function: An **exponential growth or decay function** is a function that grows or shrinks at a constant percent growth rate. The equation can be written in the form

$$\boxed{f(x) = a \cdot b^x}$$

$1+r=b$

or

$$\boxed{f(x) = a \cdot (1+r)^x}$$

Notation:

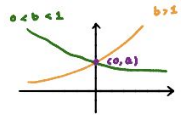
- a is the initial or starting value of the function $f(0) = c$
- r is the percent growth or decay rate, written as a decimal
- $b > 0$ is the growth factor or growth multiplier. Since powers of negative numbers behave strangely, we limit b to positive values.

Properties of the Exponential Function:

a) Domain : $\mathbb{R}$, continuity

b) Range : $(0, +\infty)$

c) Monotonicity : - $b>1$, increasing
- $0<b<1$, decreasing



d) $e^0 = 1$

e) $y=0$ is a Horizontal Asymptote

Example 3.1. India's population was 1.14 billion in the year 2008 and is growing by about 1.34 % each year. Write an exponential function for Indias population, and use it to predict the population in 2020.

Answer: – the initial value : the population was $\underbrace{1.14}_{a}$ billion in the year 2008

– the growth rate : growing by $\underbrace{1.34\%}_{r}$ per year

using the basic formula for exponential growth $f(x) = a(1+r)^x$

$f(t) = 1.14(1+0.0134)^t$

To estimate the population in 2020, $t = 2020 - 2008 = 12$

$\therefore f(12) = 1.14(1+0.0134)^{12} \approx$ $\boxed{1.337 \text{ billion people in } 2020}$

Application: **Compounded interest**

Note: We will skip this part in class you just need to know the formula of the compound interest.

If an amount A_0 is deposited into a bank account with annual interest rate r compounded times a year, then the balance after t years is

$$\boxed{A_{n,r}(t) = A_0(1+\frac{r}{n})^{nt}}$$

When n approaches infinity, we can use the limit to be an approximation of the balance after t years with a infinite n. In this case, we say that the interest is **compounded continuously.**

If the interest is continuously compounded, the balance after t years is

$$A_r(t) = A_{\infty,r}(t) = A_0 \lim_{n\to\infty}(1+\frac{r}{n})^{nt} = A_0 \lim_{n\to\infty}(1+\frac{1}{n/r})^{\frac{n}{r}rt} = A_0(\lim_{n\to\infty}(1+\frac{1}{n/r})^{\frac{n}{r}})^{rt} = \boxed{A_0e^{rt}}$$

where r is the annual interest rate.

Continuous Growth Formula

Continuous Growth can be calculated using the formula

$f(x) = ae^{rx}$

where

a is the starting amount

r is the continuous growth rate

Example 5

Radon-222 decays at a continuous rate of 17.3% per day. How much will 100mg of Radon-222 decay to in 3 days?

Since we are given a continuous decay rate, we use the continuous growth formula. Since the substance is decaying, we know the growth rate will be negative: r = -0.173

$f(3) = 100e^{-0.173(3)} \approx 59.512$ mg of Radon-222 will remain.

LECTURE 3

LOGARITHMIC FUNCTIONS

Logarithm Equivalent to an Exponential

The logarithm (base b) function, written $\log_b(x)$, is the inverse of the exponential function (base b), b^x.

This means the statement $b^a = c$ is equivalent to the statement $\log_b(c) = a$.

Properties of Logs: Inverse Properties

$\log_b(b^x) = x$

$b^{\log_b x} = x$

$\log_a a = 1$	$\log_a 1 = 0$
$\log_a x + \log_a y = \log_a xy$	$\log_a x - \log_a y = \log_a \frac{x}{y}$
$\log_a x^p = p \log_a x$	change of base: $\log_a x = \frac{\log_b x}{\log_b a}$

Graphical Features of the Logarithm: Graphically, in the function $g(x) = \log_a(x)$

- The graph has a horizontal intercept at $(1, 0)$
- The graph has a vertical asymptote at $x = 0$
- The graph is increasing and concave down $\Leftrightarrow$ for $a > 1$; decreasing & concave up for $0 < a < 1$
- The domain of the function is $x > 0$
- The range of the function is all real numbers $(-\infty, \infty)$

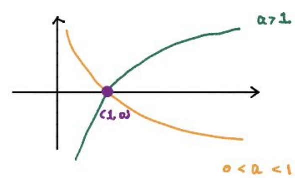
Properties of the Logarithmic Function:

a) Domain : $x > 0$ and continuity

b) Range : $\mathbb{R}$

c) monotoncity : - when $a > 1$ increasing

- when $0 < a < 1$ decreasing



d) $\log_a 1 = 0$

e) $x = 0$ is a vertical Asymptote as $x \to 0^+$, $f(x) \to \pm\infty$

Example 1

Write these exponential equations as logarithmic equations:

a) $2^3 = 8$ b) $5^2 = 25$ c) $10^{-4} = \dfrac{1}{10000}$

a) $2^3 = 8$ is equivalent to $\log_2(8) = 3$

b) $5^2 = 25$ is equivalent to $\log_5(25) = 2$

c) $10^{-4} = \dfrac{1}{10000}$ is equivalent to $\log_{10}\left(\dfrac{1}{10000}\right) = -4$

Example 2

Solve $2^x = 10$ for x.

By rewriting this expression as a logarithm, we get $x = \log_2(10)$

Example 3

Evaluate $\log(1000)$ using the definition of the common log.

To evaluate $\log(1000)$, we can say $x = \log(1000)$, then rewrite into exponential form using the common log base of 10.

$10^x = 1000$

From this, we might recognize that 1000 is the cube of 10, so $x = 3$.

We also can use the inverse property of logs to write $\log_{10}\left(10^3\right) = 3$

Values of the common log

number	number as exponential	log(number)
1000	10^3	3
100	10^2	2
10	10^1	1
1	10^0	0
0.1	10^{-1}	-1
0.01	10^{-2}	-2
0.001	10^{-3}	-3

Properties of Logs: Exponent Property

$\log_b\left(A^r\right) = r\log_b\left(A\right)$

Solving exponential equations:

Isolate the exponential expressions when possible

Take the logarithm of both sides

Utilize the exponent property for logarithms to pull the variable out of the exponent

Use algebra to solve for the variable.

Example 5

In the last section, we predicted the population (in billions) of India t years after 2008 by using the function $f(t) = 1.14(1+0.0134)^t$. If the population continues following this trend, when will the population reach 2 billion?

We need to solve for the t so that $f(t) = 2$

$2 = 1.14(1.0134)^t$ Divide by 1.14 to isolate the exponential expression

$\frac{2}{1.14} = 1.0134^t$ Take the logarithm of both sides of the equation

$\ln\left(\frac{2}{1.14}\right) = \ln\left(1.0134^t\right)$ Apply the exponent property on the right side

$\ln\left(\frac{2}{1.14}\right) = t\ln\left(1.0134\right)$ Divide both sides by ln(1.0134)

$$t = \frac{\ln\left(\frac{2}{1.14}\right)}{\ln\left(1.0134\right)} \approx 42.23 \text{ years}$$

If this growth rate continues, the model predicts the population of India will reach 2 billion about 42 years after 2008, or approximately in the year 2050.

Example 3.6. In chemistry, pH is a measure of the acidity or basicity of a liquid. The pH is related to the concentration of hydrogen ions, $[H^*]$, measured in moles per liter, by the equation

$$pH = -\log[H^*]$$

$\underset{\text{Input}}{H^*} \to \underset{\text{process}}{-\log} \to \underset{\text{output}}{pH}$

If a liquid has concentration of 0.0001 moles per liber, determine the pH. Determine the hydrogen ion concentration of a liquid with pH of 7.

Answer:

- The initial value : $PH = -\log(0.0001) = -\log(10^{-4}) = -(-4) = 4$

- Determine the hydrogen ion concentration of a liquid with PH of 7

$$7 = -\log[H^+]$$

$$-7 = \log[H^+]$$

$$10^{-7} = \cancel{10}^{\cancel{\log}[H^+]}$$

$$\boxed{H^+ = 10^{-7} = 0.0000001 \text{ moles per liter}}$$

Example 3.7. Find the domain of the function $f(x) = \log(5 - 2x)$

$\log_{10}$

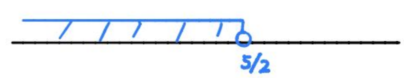
Answer:

The logarithm is only defined when the input is positive $\therefore\ 5-2x > 0$

Solving this inequality $5 - 2x > 0$

$$5 > 2x$$

$$\frac{5}{2} > x$$



$\boxed{\therefore \text{ The domain of the function is } x < \frac{5}{2}, \text{ or in interval notation } (-\infty, \frac{5}{2})}$

LECTURE 4
LIMITS

Example 4.4. (Direct substitution rule:) $\lim_{x\to 2} \frac{3x-2}{x+4}$

Answer: $\lim_{x\to 2} \frac{3x-2}{x+4} = \frac{3\cdot 2-2}{2+4} = \frac{2}{3}$

Domain is $\mathbb{R} - \{-4\}$

or $(-\infty, -4) \cup (-4, +\infty)$

- $\frac{3x-2}{x+4}$ is a rational function and its domain is $(-\infty, -4) \cup (-4, +\infty)$
- The real value 2 is in its domain so we can use 'direct substitution' rule

Example 4.5. (Direct substitution rule:) $\lim_{x\to 5} e^{2x+1}$

Answer: $\lim_{x\to 5} e^{2x+1} = e^{2\cdot 5+1} = e^{11}$

exponential function

Domain is $\mathbb{R}$

Example 4.6. (Direct substitution rule:) $\lim_{x\to 2^+}\sqrt{2x-4}$

one-sided limit

Answer:

Domain is $x \geq 2$ Since $2x-4 \geq 0$

$\lim_{x\to 2^+} \sqrt{2x-4} = \sqrt{2\cdot 2^+ - 4} = 0$

Example 4.7. Consider the rational function $f(x) = \frac{2x^2-2x}{x-1}$ and the limit $\lim_{x\to 1} f(x)$

What happens if we just plug $x = 1$ into $f(x)$?

$f(x)$ is undefined $\Leftarrow$ denominator would be zero when $x = 1$

Limit Laws

If $\lim_{x \to a} f(x) = M$ and $\lim_{x \to a} g(x) = N$

Sum	$\lim_{x \to a} \left[f(x) + g(x) \right] = M + N$
Difference	$\lim_{x \to a} \left[f(x) - g(x) \right] = M - N$
Constant	$\lim_{x \to a} \left[k \bullet f(x) \right] = k \bullet M$
Product	$\lim_{x \to a} \left[f(x)\, g(x) \right] = M \bullet N$
Quotient	$\lim_{x \to a} \left[\frac{f(x)}{g(x)} \right] = \frac{M}{N}$ $\quad N \neq 0$
Power	$\lim_{x \to a} \left[f(x) \right]^n = M^n$ $\quad$ *n* is a positive integer
Root	$\lim_{x \to a} \left[\sqrt[n]{f(x)} \right] = \sqrt[n]{M}$ $\quad$ *n* is a positive integer

ARITHMETIC SERIES

$$\sum_{i=0}^{n-1}(a+id) = \frac{n(2a+(n-1)d)}{2}$$

Example 4.13. Find $\sum_{i=2}^{5}(1+2i)$ $= (1+2\cdot 2) + (1+2\cdot 3) + (1+2\cdot 4) + (1+2\cdot 5)$

The starting point $a = 1+2\cdot 2 = 5$

The common difference $d = 2$

The # of terms is $n = 4$

$$\sum_{i=2}^{5}(1+2i) = \frac{4(2\cdot 5+(4-1)\cdot 2)}{2} = 32$$