Final exam (2014)

1. (a) [7] If $W = \{(a_1, a_2, a_3, a_4) \in \mathbb{R}^4 : \; a_1 + a_3 = 0, a_1 - a_2 + 4a_4 = 0\}$, is W a subspace of $\mathbb{R}^4$? If yes, find a basis of W.

   (b) [7] If $T : \mathbb{R}^3 \to \mathbb{R}^2, \; T(a, b, c) = (a + b, a - b)$; determine whether T is $1 - 1$ and onto.

2. (a) [7] Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by $T(-1, 1) = (1, 2, 3)$ and $T(1, 1) = (0, 1, 2)$. Assuming that T is a linear transformation, find $T(0, 1)$.

   (b) [7] Let $T : M_{3\times2}(\mathbb{R}) \to M_{2\times3}(\mathbb{R})$ be a linear transformation defined by

$$T\begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & a \end{pmatrix},$$

   find bases for $R(T)$ and $N(T)$, and verify the dimension theorem.

3. (a) [7] Let $T : P_2(\mathbb{R}) \to \mathbb{R}^4$ be a linear transformation defined by $T(a+bx+c^2) = (a+b, b+a, a+c, b+c)$. Find the matrix that represents T relative to the basis $B = \{1, 1 + x, 1 + x + x^2\}$ of $P_2(\mathbb{R})$ and the basis $C = \{(1, 1, 1, 1), (1, 0, 0, 0), (1, 1, 0, 0), (0, 0, 1, 0)\}$ of $\mathbb{R}^4$. ($P_2(\mathbb{R})$ is the vector space of polynomials of degree ≤ 2).

   (b) [7] If $W_1 = \left\{\begin{pmatrix} a & a \\ a & b \end{pmatrix} : \; a, b \in \mathbb{R}\right\}$ and $W_2 \left\{\begin{pmatrix} c & c \\ d & c \end{pmatrix} : \; c, d \in \mathbb{R}\right\}$, find bases of $W_1 + W_2$ and $W_1 \cap W_2$.

4. (a) [7] Let $T : V \to W$ be a $1-1$ linear transformation. If $S = \{v_1, v_2, \ldots, v_n\}$ is linearly independent, show that $T(S) = \{T(v_1), T(v_2), \ldots, T(v_n)\}$ is also linearly independent.

   (b) [9] If $T : \mathbb{R}^2 \to \mathbb{R}^2$, $T(a,b) = (3a+4b, 3a-4b)$. Let $B = \{(1,1),(0,1)\}$ and $C = \{(1,1),(1,3)\}$, find $[T]_B$, $[T]_C$ and show $[T]_C = Q^{-1}[T]_B Q$ for some invertible matrix Q.

5. (a) [7] Find the rank of

$$A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 1 & -2 \\ 1 & 5 & 4 \end{pmatrix}$$

   and its inverse, if it exists.

   (b) [7] Let $T : P_2(\mathbb{R}) \to \mathbb{R}^3$ is defined by $T(a+bx+cx^2) = (a+b, b+c, c+a)$. Is T invertible? If yes, find T^{-1}.

6. (a) [7] Let $V = \{(a_1, a_2, a_3, a_4) : a_1 - 2a_2 + 3a_3 - 4a_4 = 0\}$. Show that $(2,0,2,2) \in V$ and extend $(2,0,2,2)$ to a basis of V.

   (b) [7]If $V = P_2(\mathbb{R})$, $T(f(x)) = xf'(x) + f(2)x + f(3)$, find the eigenvalues of T and a basis B for V such that $[T]_B$ is a diagonal matrix.

7. (a) [7] Let $A = \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}$. Find A^{100}.

   (b) [7] Let A be $n \times n$. Let λ be an eigenvalue of A. Show that λ is also an eigenvalue of A^{T}, the transpose of A. Show by example that, if E_λ is the eigenspace of A and E'_λ, the eigenspace of A^{T}, then $E_\lambda \neq E'_\lambda$.

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