1. Let $U = \{(x, y, z) \in \mathbb{R}^3 \mid x+y+z = 0\}$. Which one of the following statements is true? (1)

`mark (X) the correct answer:`

- [A] U is not a subspace of $\mathbb{R}^3$
- [B] U is a subspace of $\mathbb{R}^3$ and $\dim U = 3$
- [**C**] U is a subspace of $\mathbb{R}^3$ and $\{(1, 1, -2), (0, 1, -1)\}$ is a basis of U
- [D] U is a line in $\mathbb{R}^3$ with direction vector $(1, 1, 1)$
- [E] U is a subspace of $\mathbb{R}^3$ and $\{(1, 0, -1), (-1, 0, 1)\}$ is a basis of U
- [F] U is a plane in $\mathbb{R}^3$ with normal vector $(-1, 2, -1)$

Solution: As U is a plain which contains $(0, 0, 0)$, it is a linear subspace of $\mathbb{R}^3$ of dimension 2. So one excludes [A], [B] and [D]. Since $\{(1, 0, -1), (-1, 0, 1)\}$ is dependent, it is not a basis of U, so one excludes [E]. The normal vector to U is collinear to $(1, 1, 1)$, so one excludes [F]. Hence, the correct answer is [C].

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2. Which two of the following statements are true? (1)

I. $\{1, \sin^2 x, \cos^2 x\}$ is a linearly independent set of vectors in the vector space of all real-valued functions $F(\mathbb{R}) = \{f \mid f : \mathbb{R} \to \mathbb{R}\}$

II. A homogeneous system of linear equations is always consistent

III. If A and B are 3×3 matrices, and both A and B are invertible, then their product AB is also an invertible matrix

IV. If u and v are linearly dependent vectors in $\mathbb{R}^3$, then $\dim(Span\{u, v\}) = 2$.

`mark (X) the correct answer:`

- [A] I. and II.
- [B] I. and III.
- [C] I. and IV.
- [**D**] II. and III.
- [E] II. and IV.
- [F] III. and IV.

Solution: I. [False] Since $\sin^2 x + \cos^2 x = 1$, these vectors (functions) are linearly dependent.

II. [True] A homogeneous system has always a zero solution.

III. [True] We have $(AB)^{-1} = B^{-1}A^{-1}$.

IV. [False] Since $\{u, v\}$ are dependent, $\dim(Span\{u, v\}) = 1$.

So the correct answer is [D].

3. Let A be a square $n \times n$ matrix with $n \geq 2$.

Which of the following statements are true? (1)

I. If $rank(A) = 1$, there is just one parameter in the general solution of the system $Ax = 0$

II. If $rank(A) = 1$, there are $n - 1$ parameters in the general solution of the system $Ax = 0$

III. If A is invertible, the homogeneous system $Ax = 0$ has infinitely many solutions

IV. If the system $Ax = 0$ has infinitely many solutions, then $rank(A) < n$

`mark (X) the correct answer`:

- [A] I. only
- [B] II. only
- [C] I. and III.
- [D] II. and III.
- [E] I. and IV.
- [**F**] II. and IV.

Solution: Using the formula $rank(A) + \#parameters = \#columns$, we obtain that if $rank(A) = 1$, then $\#parameters = n - 1$, so II. is correct. Also IV. is correct by the same formula. Hence, the correct answer is [F].

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4. Find the polar form of the complex number $\frac{1-\sqrt{3}i}{i-1}$. (see the last page for the table of trigonometric functions) (1)

`mark (X) the correct answer`:

- [A] $\sqrt{2}(\cos(-7\pi/12) + i\sin(-7\pi/12))$
- [B] $\sqrt{2}(\cos(5\pi/12) + i\sin(5\pi/12))$
- [C] $\sqrt{2}(\cos(-\pi/12) + i\sin(-\pi/12))$
- [D] $\sqrt{2}(\cos(\pi/12) + i\sin(\pi/12))$
- [E] $\sqrt{2}(\cos(-5\pi/12) + i\sin(-5\pi/12))$
- [**F**] $\sqrt{2}(\cos(11\pi/12) + i\sin(11\pi/12))$

Solution: We have

$$\frac{1-\sqrt{3}i}{i-1} = \frac{\frac{1}{2} - \frac{\sqrt{3}}{2}i}{-\frac{1}{2} + \frac{1}{2}i} = \frac{e^{i(-\pi/3)}}{\frac{1}{\sqrt{2}}e^{i(3\pi/4)}} = \sqrt{2}e^{i(-\pi/3-3\pi/4)} = \sqrt{2}e^{i(-13\pi/12)} = \sqrt{2}e^{i(11\pi/12)}$$

The correct answer is [F].

5. Let $B = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$ and consider the subset $U = \{A \in M_{2\times 2}(\mathbb{R}) \mid BA = -AB\}$.

Which one of the following statements is true? (1)

`mark (X) the correct answer:`

[A] U is not a subspace of the vector space of 2×2 matrices $M_{2\times 2}(\mathbb{R})$

[B] U is a subspace of $M_{2\times 2}(\mathbb{R})$, and $\dim U = 0$

[C] U is a subspace of $M_{2\times 2}(\mathbb{R})$, and $\dim U = 1$

[D] U is a subspace of $M_{2\times 2}(\mathbb{R})$, and $\dim U = 2$

[E] U is a subspace of $M_{2\times 2}(\mathbb{R})$, and $\dim U = 3$

[F] U is a subspace of $M_{2\times 2}(\mathbb{R})$, and $\dim U = 4$

Solution: Suppose $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $BA = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a-c & b-d \\ 0 & 0 \end{pmatrix}$. On the other hand $-AB = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -a & a \\ -c & c \end{pmatrix}$. Hence, $AB = -BA$ would imply that $c = 0$, $a = -a$ and $b - d = a$.

So $U = \{\begin{pmatrix} 0 & b \\ 0 & b \end{pmatrix} \mid b \in \mathbb{R}\}$ is a subspace of $M_{2\times 2}(\mathbb{R})$ of dimension 1.

The correct answer is [C].

---

6. If $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}$, then the third row of A^{-1} is: (1)

`mark (X) the correct answer:`

[A] $(0 \;\; 1 \;\; 0)$

[B] $(1 \;\; 0 \;\; 2)$

[C] $(1 \;\; 0 \;\; -1)$

[D] $(0 \;\; 0 \;\; 1)$

[E] $(1 \;\; 0 \;\; 1)$

[F] $(-1 \;\; 1 \;\; 1)$

Solution: Let (x_1, x_2, x_3) denote the third row of A^{-1}. Since $A^{-1}A = I$, we obtain

$$\begin{pmatrix} * & * & * \\ * & * & * \\ x_1 & x_2 & x_3 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The correct answer is [F].

7. For a non-homogeneous system of 13 equations in 15 unknowns, answer the following three questions (Yes/No): (1)

- Can the system be inconsistent?
- Can the system have a unique solution?
- Can the system have infinitely many solutions?

`mark (X) the correct answer:`

[A] Yes, Yes, Yes

[B] Yes, Yes, No

[C] No, Yes, Yes

[D] Yes, No, Yes

[E] No, No, Yes

[F] Yes, No, No

Solution: The answers to the first and the third questions are YES.

If system has a unique solution, then the number of leading ones in its REF (≤ 13) must coincide to the number of unknowns (15).

So the correct answer is [D].

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8. Let A be a square $n \times n$ matrix.

Which one of the statements below is **not equivalent** to the statement (1)

"The columns of A are linearly independent"

`mark (X) the correct answer:`

[A] The rows of A are linearly independent

[B] $rank(A) = n$

[C] $\det(A) \neq 1$

[D] The rows of A form a basis of $\mathbb{R}^n$

[E] The homogeneous system $Ax = 0$ has a unique solution

[F] A is invertible

Solution: The correct answer is [C].

9. Let $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ for some numbers $a, b, c, d, e, f, g, h, i$ be such that $\det(A) = 2$.

Using the elementary row/column operations compute the determinant of the matrix

$$\begin{pmatrix} b+5c & e+5f & h+5i \\ 3a & 3d & 3g \\ -2c & -2f & -2i \end{pmatrix}$$

(1)

`mark (X) the correct answer:`

- A -64
- B -18
- C -12
- **D** 12
- E 18
- F 64

Solution: We apply elementary row operations to the matrix A: First, we switch the first and the second rows (the determinant changes its sign). Next, we multiply the first row by 3, we add to the second row the third row multiplied by 5 and, finally, we multiply the third row by -2. As a result we get a matrix of determinant 12 which is the transposed to the one in question.

The correct answer is D.

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10. The vectors $u_1 = (1, -1, 2)$, $u_2 = (-5, -1, 2)$, and $u_3 = (0, 2, 1)$ form an orthogonal basis of $\mathbb{R}^3$. So that any vector $v \in \mathbb{R}^3$ can be expressed in a unique way as a linear combination $v = a_1u_1 + a_2u_2 + a_3u_3$, where $a_1, a_2, a_3 \in \mathbb{R}$ are coordinates of v with respect to the basis $\{u_1, u_2, u_3\}$. Find the coordinate a_2 for $v = (1, 0, 1)$. (1)

`mark (X) the correct answer:`

- A -0.2
- **B** -0.1
- C -1
- D 1
- E 0.1
- F 0.2

Solution: We have

$$a_2 = \frac{v \cdot u_2}{||u_2||^2} = \frac{-3}{30} = -\frac{1}{10}$$

So the correct answer is B.

11. Let $U = Span\{(-1,1,1,0),(1,0,1,1),(2,1,4,3),(0,1,2,2)\}$ in $\mathbb{R}^4$.

(a) Find a basis for U which is a subset of the given spanning set above. (1)

ANSWER:

Solution: $\{(-1,1,1,0),(1,0,1,1),(0,1,2,2)\}$ is a basis.

(b) Extend the basis found in part (a) to a basis of $\mathbb{R}^4$. (1)

ANSWER:

Solution: If we add $(0,0,1,0)$ to the basis above, we obtain a basis of $\mathbb{R}^4$.

12. Let $W = \{(x, y, z, w) \in \mathbb{R}^4 \mid y - z - w = 0\}$

(a) Find a basis for W. (1)

ANSWER:

Solution: $W = Null(A)$, where $A = (0\ 1\ \ -1\ \ -1)$. We obtain

$$W = \{(r, s + t, s, t) \in \mathbb{R}^4 \mid r, s, t \in \mathbb{R}\},$$

hence $\{(0, 1, 1, 0), (0, 1, 0, 1), (1, 0, 0, 0)\}$ is a basis for W.

(b) Use the Gram-Schmidt algorithm to find an orthogonal basis for W. (1)

ANSWER:

Solution: $\{(0, 1, 1, 0), (0, 1, -1, 2), (1, 0, 0, 0)\}$ is an orthogonal basis of W.

(c) Find the best approximation of $(0, 1, 1, 1)$ by a vector in W. (1)

ANSWER:

Solution: $proj_W(0, 1, 1, 1) = (0, 4/3, 2/3, 2/3)$.

13. Let $A = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$

(a) Find the characteristic polynomial of A. (1)

ANSWER:

Solution:

$$\det(A - \lambda I) = \begin{pmatrix} -1-\lambda & 1 & 1 \\ 1 & -1-\lambda & 1 \\ 1 & 1 & -1-\lambda \end{pmatrix} = -(1+\lambda)^3 + 1 + 1 + 3(1+\lambda)$$

$$= -\lambda^3 - 3\lambda^2 + 4 = (\lambda+2)^2(1-\lambda)$$

(b) Using the characteristic polynomial explain why the eigenvalues of A are -2 and 1. (1/2)

Solution: The roots of the characteristic polynomial that are -2 and 1 are the eigenvalues of A.

(c) Find a basis of the eigenspace $E_{-2} = \{v \in \mathbb{R}^3 \mid Av = -2v\}$. (1)

ANSWER:

Solution:

$$E_{-2} = \ker(A + 2I) = \ker \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \ker \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \{(-s - t, s, t) \mid s, t \in \mathbb{R}\}$$

So that $\{(-1, 1, 0), (-1, 0, 1)\}$ is a basis for E_{-2}.

(d) Find a basis of the eigenspace $E_1 = \{v \in \mathbb{R}^3 \mid Av = v\}$. (1)

ANSWER:

Solution:

$$E_1 = \ker(A - I) = \ker \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} = \{(s, s, s) \mid s \in \mathbb{R}\}$$

So that $\{(1, 1, 1)\}$ is a basis for E_1.

(e) Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$. (1/2)

ANSWER: $P =$ $D =$

Solution: $P = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

14. State whether each of the following is (always) true, or is (possibly) false, in the respective box. Provide the respective justification/example.

(a) If A is an 5×4 matrix and if a row echelon form of A has a row of zeros, then $rank(A) < 4$

ANSWER (True/False): (1/2)

Justification/example: (1/2)

False (Can have rank 4)

(b) The dimension of the null space of the 1×4 matrix $(-1\ 0\ 1\ 2)$ is 3

ANSWER (True/False): (1/2)

Justification/example: (1/2)

True (The dimension of the null space is $4 - (rank) = 4 - 1 = 3$)

(c) Let z_1 and z_2 be two complex numbers which are not real numbers. Then their product $z_1 \cdot z_2$ can not be a real number.

ANSWER (True/False): ☐ (1/2)

Justification/example: (1/2)

False ($i \cdot i = -1$)

(d) The function $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (x, x + y)$ is a linear transformation.

ANSWER (True/False): ☐ (1/2)

Justification/example: (1/2)

True (Check the criteria for being a linear transformation)

15. (BONUS question) Define a linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ by

$$T(\begin{pmatrix} x \\ y \\ z \end{pmatrix}) = \begin{pmatrix} x-y \\ y-z \\ z-x \end{pmatrix}$$

(a) Find the standard matrix of T. (1)

ANSWER:

Solution: We have $T(e_1) = e_1 - e_3$, $T(e_2) = -e_1 + e_2$, $T(e_3) = -e_2 + e_3$. So

$$A_T = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix}.$$

(b) Find a basis for the image of T. (1)

ANSWER:

Solution: $im(T) = Span\{T(e_1), T(e_2), T(e_3)\}$. Since $T(e_1)$, $T(e_2)$ are linearly independent and $T(e_3) = -T(e_1) - t(e_2)$, $\{T(e_1), T(e_2)\}$ form a basis of $im(T)$.

The last page (use it for draft computations only).

Anything written on the last page will not be graded or taken into account while grading.

$$\sin(\tfrac{\pi}{6}) = \cos(\tfrac{\pi}{3}) = \tfrac{1}{2}, \quad \sin(\tfrac{\pi}{3}) = \cos(\tfrac{\pi}{6}) = \tfrac{\sqrt{3}}{2}$$
$$\sin(\tfrac{\pi}{4}) = \cos(\tfrac{\pi}{4}) = \tfrac{\sqrt{2}}{2}, \quad \sin(0) = \cos(\tfrac{\pi}{2}) = 0, \quad \sin(\tfrac{\pi}{2}) = \cos(0) = 1.$$