4.4 Summary—Important Facts about Determinants

3. Evaluate the determinant of the following matrices in the manner indicated.

(h) $\begin{bmatrix} 1 & -1 & 2 & -1 \\ -3 & 4 & 1 & -1 \\ 2 & -5 & -3 & 8 \\ -2 & 6 & -4 & 1 \end{bmatrix}$
along the fourth row

4. Evaluate the determinant of the following matrices by any legitimate method.

(h) $\begin{bmatrix} 1 & -2 & 3 & -12 \\ -5 & 12 & -14 & 19 \\ -9 & 22 & -20 & 31 \\ -4 & 9 & -14 & 15 \end{bmatrix}$

5.1 Eigenvalues and Eigenvectors

3. For each of the following linear operators T on a vector space V and ordered bases β, compute $[T]_\beta$, and determine whether β is a basis consisting of eigenvectors of T.

(d) $V = P_2(\mathbb{R})$,

$$T(a + bx + cx^2) = (-4a + 2b - 2c) - (7a + 3b + 7c)x + (7a + b + 5c)x^2,$$

and $\beta = \{x - x^2, -1 + x^2, -1 - x + x^2\}$.

4. For each of the following matrices $A \in M_{n\times n}(\mathbb{F})$,

(i) Determine all the eigenvalues of A.

(ii) For each eigenvalue λ of A, find the set of eigenvectors corresponding to λ.

(iii) If possible, find a basis for $\mathbb{F}^n$ consisting of eigenvectors of A.

(iv) If successful in finding such a basis, determine an invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$.

(b) $A = \begin{bmatrix} 0 & -2 & -3 \\ -1 & 1 & -1 \\ 2 & 2 & 5 \end{bmatrix}$ for $\mathbb{F} = \mathbb{R}$

(d) $A = \begin{bmatrix} 2 & 0 & -1 \\ 4 & 1 & -4 \\ 2 & 0 & -1 \end{bmatrix}$ for $\mathbb{F} = \mathbb{R}$

5. For each linear operator T on V, find the eigenvalues of T and an ordered basis β for V such that $[T]_\beta$ is a diagonal matrix.

(c) $V = \mathbb{R}^3$ and $T(a, b, c) = (-4a + 3b - 6c, 6a - 7b + 12c, 6a - 6b + 11c)$

(d) $V = P_1(\mathbb{R})$ and $T(ax + b) = (-6a + 2b)x + (-6a + b)$

(g) $V = P_3(\mathbb{R})$ and $T(f(x)) = xf'(x) + f''(x) - f(2)$

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(a) Let T be a linear operator on a vector space V, and let x be an eigenvector of T corresponding to the eigenvalue λ. For any positive integer m, prove that x is an eigenvector of T^m corresponding to the eigenvalue λ^m.

(b) State and prove the analogous result for matrices.