(A)

MAT 2384 C
DIFFERENTIAL EQUATIONS AND NUMERICAL METHODS
MIDTERM
March 4, 2015

Instructor: Dr. Steve Desjardins

Duration: 80 minutes

Name: Solutions

Student Number:

Instructions:
- Print your name and student number on this page.
- Verify that your copy of the exam has all 6 pages.
- You must answer all questions.
- Write your answers in the spaces below the questions. You may use the backs of the pages if necessary.
- **No Notes or Books.**
- **Basic scientific calculators only - graphing and/or programmable calculators are NOT permitted.**

A

Question 1 (6 marks) Solve the initial value problems:

(a) $e^y y' = \sec^2 x, \quad y(0) = 0$

this DE is separable $\quad e^y \dfrac{dy}{dx} = \sec^2 x$

or $\quad e^y\,dy = \sec^2 x\,dx$

integrate $\quad \int e^y\,dy = \int \sec^2 x\,dx + C$

we get $\quad e^y = \tan x + C$

so $\quad y = \ln|\tan x + C| \quad$ (general solution)

$y(0) = 0 \Rightarrow 0 = \ln(\tan(0) + c) = \ln(c) \Rightarrow c = 1$

$\therefore$ the unique solution is $\boxed{y = \ln|1 + \tan x|}$

(b) $y' + \dfrac{y}{x} = \dfrac{2x}{y}, \quad y(1) = 3$

this is a Bernoulli equation with $p(x) = \frac{1}{x}$, $g(x) = 2x$, $a = -1$

so we let $u(x) = (y(x))^{1-a} = (y(x))^2$ and the DE

becomes $\quad u' + \dfrac{2}{x} = 4x \quad$ (linear with $f(x) = \frac{2}{x}$, $r(x) = 4x$)

so $\mu(x) = e^{\int \frac{2}{x}dx} = e^{2\ln x} = x^2$

then $\quad u(x) = \dfrac{1}{x^2}\left[\int (x^2)(4x)\,dx + C\right]$

$= x^{-2}\left(\int 4x^3\,dx + C\right)$

$= x^{-2}(x^4 + C) = x^2 + Cx^{-2}$

and so $y^2 = x^2 + Cx^{-2} \quad$ (general solution)

then $y(1) = 3 \Rightarrow 9 = 1 + C \Rightarrow C = 8$ and so the

unique solution is $\boxed{y = \sqrt{x^2 + 8x^{-2}}}$

Question 2 (6 marks) Solve the initial value problem:

$$(2e^x y^2 + 6xy + 7y^3)\,dx + (8e^x y + 9x^2 + 35xy^2)\,dy = 0, \quad y(0) = 0\,.$$

$M(x,y) = 2e^x y^2 + 6xy + 7y^3 \Rightarrow M_y = 4e^x y + 6x + 21y^2$

$N(x,y) = 8e^x y + 9x^2 + 35xy^2 \Rightarrow N_x = 8e^x y + 18x + 35y^2$

$\Big\}\ M_y \neq N_x$ DE not exact

$$\frac{M_y - N_x}{M} = \frac{-4e^x y - 12x - 14y^2}{2e^x y^2 + 6xy + 7y^3} = \frac{-2(2e^x y + 6x + 7y^2)}{y(2e^x y + 6x + 7y^2)} = \frac{-2}{y} \quad \left(\text{function of } y \text{ only}\right)$$

so $\mu(y) = e^{-\int -\frac{2}{y}dy} = e^{2\ln y} = y^2$ and the DE becomes

$$(2e^x y^4 + 6xy^3 + 7y^5)\,dx + (8e^x y^3 + 9x^2 y^2 + 35xy^4)\,dy = 0$$

$M^*(x,y) = 2e^x y^4 + 6xy^3 + 7y^5 \Rightarrow M_y^* = 8e^x y^3 + 18xy^2 + 35y^4$

$N^*(x,y) = 8e^x y^3 + 9x^2 y^2 + 35xy^4 \Rightarrow N_x^* = 8e^x y^3 + 18xy^2 + 35y^4$

$\Big\}\ M_y^* = N_x^*$ DE exact

$$F(x,y) = \int M^*(x,y)\,dx + g(y) = \int (2e^x y^4 + 6xy^3 + 7y^5)\,dx + g(y)$$
$$= 2e^x y^4 + 3x^2 y^3 + 7xy^5 + g(y)$$

then $\frac{\partial F}{\partial y} = 8e^x y^3 + 9x^2 y^2 + 35xy^4 + g'(y) = N^*(x,y) = 8e^x y^3 + 9x^2 y^2 + 35xy^4$

so $g'(y) = 0 \Rightarrow$ take $g(y) = 0$

and thus $F(x,y) = 2e^x y^4 + 3x^2 y^3 + 7xy^5$

and the general solution is $2e^x y^4 + 3x^2 y^3 + 7xy^5 = C$

$y(0) = 0 \Rightarrow 2e^0(0) + 3(0)(0) + 7(0)(0) = C \Rightarrow C = 0$

$\therefore$ the unique solution is $\boxed{2e^x y^4 + 3x^2 y^3 + 7xy^5 = 0}$

A

Question 3 (6 marks) Solve the initial value problems:

(a) $y'' - 14y' + 49y = 0,\ \ y(0) = 0,\ \ y'(0) = 3$

the characteristic equation is $\lambda^2 - 14\lambda + 49 = (\lambda - 7)^2 = 0$

so the roots are $\lambda_1 = \lambda_2 = 7$ and the general solution

is $y(x) = C_1 e^{7x} + C_2 x e^{7x}$

$y(0) = 0 \Rightarrow 0 = C_1 e^0 + C_2(0)e^0 \Rightarrow C_1 = 0$

$y'(x) = 7C_1 e^{7x} + C_2 e^{7x} + 7C_2 x e^{7x}$

$y'(0) = 3 \Rightarrow 3 = 7C_1 e^0 + C_2 e^0 + 7C_2(0)e^0 \Rightarrow C_2 = 3$

$\therefore$ the unique solution is $\boxed{y(x) = 3xe^{7x}}$

(b) $y'' - 2y' + 10y = 0,\ \ y(0) = 2,\ \ y'(0) = 8$

the char. eq $\lambda^2 - 2\lambda + 10 = 0 \Rightarrow \lambda_{1,2} = \dfrac{2 \pm \sqrt{(-2)^2 - 4(10)}}{2} = 1 \pm 3i$

so the general solution is $y(x) = C_1 e^x \cos(3x) + C_2 e^x \sin(3x)$

$y(0) = 2 \Rightarrow 2 = C_1 e^0 \cos(0) + C_2 e^0 \sin(0) \Rightarrow C_1 = 2$

$y'(x) = C_1 e^x \cos(3x) - 3C_1 e^x \sin(3x) + C_2 e^x \sin(3x) + 3C_2 e^x \cos(3x)$

$y'(0) = 8 \Rightarrow 8 = C_1 e^0 \cos(0) - 3C_1 e^0 \sin(0) + C_2 e^0 \sin(0) + 3C_2 e^0 \cos(0)$

$\Rightarrow C_1 + 3C_2 = 8 \Rightarrow C_2 = 2$

$\therefore$ the unique solution is $\boxed{y(x) = 2e^x \cos(3x) + 2e^x \sin(3x)}$

(A)

Question 4 (6 marks) Find the general solutions of the differential equations:

(a) $y''' - 2y'' - y' + 2y = 0$

the char. eq. is $\lambda^3 - 2\lambda^2 - \lambda + 2 = 0$

by inspection, $\lambda = 1$ is a root, so

$\lambda^3 - 2\lambda^2 - \lambda + 2 = (\lambda - 1)(\lambda^2 - \lambda - 2) = (\lambda - 1)(\lambda - 2)(\lambda + 1)$

so roots are $\lambda_{1,2} = \pm 1$, $\lambda_3 = 2$

so the general solution is $\boxed{y(x) = C_1 e^{x} + C_2 e^{-x} + C_3 e^{2x}}$

(b) $x^3 y''' + 2x^2 y'' - 4xy' + 4y = 0, \quad x > 0$

the char. eq. is $m(m-1)(m-2) + 2m(m-1) - 4m + 4 = 0$

or $m(m-1)(m-2) + 2m(m-1) - 4(m-1) = 0$

or $(m-1)[m(m-2) + 2m - 4] = 0$

or $(m-1)[m(m-2) + 2(m-2)] = 0$

or $(m-1)(m+2)(m-2) = 0$

so roots are $m_1 = 1$, $m_{2,3} = \pm 2$

and the general solution is $\boxed{y(x) = C_1 x + C_2 x^2 + C_3 x^{-2}}$

(A)

Question 5 (6 marks)
(a) Consider $f(x) = x^3 + 7x - 6$. If you were to use Fixed Point Iteration to find the root of this function in the interval $[0, 1]$, what would you use for $g(x)$ and why?

$f(x) = 0 \Rightarrow x^3 + 7x - 6 = 0 \Rightarrow 7x = 6 - x^3 \Rightarrow x = \frac{6-x^3}{7}$

so $\boxed{g(x) = \frac{6-x^3}{7}}$ satisfies fixed point condition

and $\boxed{|g'(x)| = \frac{3}{7}x^2 < 1 \text{ on } [0,1]}$ generates convergent sequence

(b) Use Newton's Method to find the root of this $f(x)$ to 4 decimal places. Start with $x_0 = 1$. Verify your answer.

$f(x) = x^3 + 7x - 6 \Rightarrow f'(x) = 3x^2 + 7$

so $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 + 7x_n - 6}{3x_n^2 + 7} = \frac{2x_n^3 + 6}{3x_n^2 + 7}$

$x_0 = 1, \quad x_1 = \frac{2+6}{3+7} = \frac{8}{10} = 0.8$

$x_2 = 0.7874 = x_3 \therefore$ stop

$\boxed{f(0.7874) \approx -1.3 \times 10^{-5}}$ so okay $\therefore$ root is $\boxed{0.7874}$

(c) If we were going to find the interpolating polynomial $p_n(x)$ for 4 data points, what is the maximum possible degree that the polynomial could have?

(A) 2 (B) 3 (C) 4 (D) 5 (E) 6 (F) impossible to answer

$(n+1 = 4)$

(B)

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B

Question 1 (6 marks) Solve the initial value problems:

(a) $e^y\, y' = \dfrac{1}{1+x^2}$, $y(0) = 0$

the DE is separable $e^y\, dy = \frac{1}{1+x^2}\, dx$

integrate $\int e^y\, dy = \int \frac{1}{1+x^2}\, dx + C$

we get $e^y = \arctan x + C$

so $y = \ln|\arctan x + C|$ (general solution)

$y(0)=0 \Rightarrow 0 = \ln|\arctan(0) + C| = \ln C \Rightarrow C = 1$

$\therefore$ the unique solution is $\boxed{y = \ln|1 + \arctan x|}$

(b) $y' + \dfrac{y}{x} = \dfrac{2x}{y}$, $y(1) = 4$

(See version A)

$y^2 = x^2 + Cx^{-2}$

$y(1) = 4 \Rightarrow 16 = 1 + C \Rightarrow C = 15$

$\therefore$ the unique solution is $\boxed{y = \sqrt{x^2 + 15x^{-2}}}$

Question 2 (6 marks) Solve the initial value problem:

$$(3e^x y + 8xy^2 + 5y^3)\,dx + (9e^x + 16x^2 y + 25xy^2)\,dy = 0, \quad y(0) = 0\,.$$

$M(x,y) = 3e^x y + 8xy^2 + 5y^3 \Rightarrow M_y = 3e^x + 16xy + 15y^2$
$N(x,y) = 9e^x + 16x^2 y + 25xy^2 \Rightarrow N_x = 9e^x + 32xy + 25y^2$ $\Big\}$ $M_y \neq N_x$ DE not exact

$$\frac{M_y - N_x}{M} = \frac{-6e^x - 16xy - 10y^2}{3e^x y + 8xy^2 + 5y^3} = \frac{-2(3e^x + 8xy + 5y^2)}{y(3e^x + 8xy + 5y^2)} = -\frac{2}{y}$$

so $\mu(y) = y^2$ and the DE becomes

$$(3e^x y^3 + 8xy^4 + 5y^5)dx + (9e^x y^2 + 16x^2 y^3 + 25xy^4)dy = 0$$

$M^*(x,y) = 3e^x y^3 + 8xy^4 + 5y^5 \Rightarrow M^*_y = 9e^x y^2 + 32xy^3 + 25y^4$
$N^*(x,y) = 9e^x y^2 + 16x^2 y^3 + 25xy^4 \Rightarrow N^*_x = 9e^x y^2 + 32xy^3 + 25y^4$ $\Big\}$ $M^*_y = N^*_x$ DE exact

$$F(x,y) = \int N^*(x,y)\,dy + g(x) = \int (9e^x y^2 + 16x^2 y^3 + 25xy^4)\,dy + g(x)$$
$$= 3e^x y^3 + 4x^2 y^4 + 5xy^5 + g(x)$$

then $\dfrac{dF}{dx} = 3e^x y^3 + 8xy^4 + 5y^5 + g'(x) = M^*(x,y) = 3e^x y^3 + 8xy^4 + 5y^5$

thus $g'(x) = 0 \Rightarrow$ take $g(x) = 0$

and so $F(x,y) = 3e^x y^3 + 4x^2 y^4 + 5xy^5$

the general solution is $3e^x y^3 + 4x^2 y^4 + 5xy^5 = C$

$y(0) = 0 \Rightarrow 3e^0(0) + 4(0)(0) + 5(0)(0)^5 = C \Rightarrow C = 0$

$\therefore$ the unique solution is $\boxed{3e^x y^3 + 4x^2 y^4 + 5xy^5 = 0}$

Question 3 (6 marks) Solve the initial value problems:

(a) $y'' - 12y' + 36y = 0, \quad y(0) = 0, \quad y'(0) = 4$

the char. eq. is $\lambda^2 - 12\lambda + 36 = (\lambda - 6)^2 = 0$

so $\lambda_1 = \lambda_2 = 6$ and the general solution is

$y(x) = C_1 e^{6x} + C_2 x e^{6x}$

$y(0) = 0 \Rightarrow 0 = C_1 e^0 + C_2 (0) e^0 \Rightarrow C_1 = 0$

$y'(x) = 6C_1 e^{6x} + C_2 e^{6x} + 6C_2 x e^{6x}$

$y'(0) = 4 \Rightarrow 4 = 6C_1 e^0 + C_2 e^0 + 6C_2 (0) e^0 \Rightarrow C_2 = 4$

$\therefore$ the unique solution is $\boxed{y(x) = 4xe^{6x}}$

(b) $y'' - 2y' + 5y = 0, \quad y(0) = 3, \quad y'(0) = 9$

the char. eq. is $\lambda^2 - 2\lambda + 5 = 0$, so $\lambda_{1,2} = \dfrac{2 \pm \sqrt{(-2)^2 - 4(5)}}{2} = 1 \pm 2i$

and the general solution is $y(x) = C_1 e^x \cos(2x) + C_2 e^x \sin(2x)$

$y(0) = 3 \Rightarrow 3 = C_1 e^0 \cos(0) + C_2 e^0 \sin(0) \Rightarrow C_1 = 3$

$y'(x) = C_1 e^x \cos(2x) - 2C_1 e^x \sin(2x) + C_2 e^x \sin(2x) + 2C_2 e^x \cos(2x)$

$y'(0) = 9 \Rightarrow 9 = C_1 e^0 \cos(0) - 2C_1 e^0 \sin(0) + C_2 e^0 \sin(0) + 2C_2 e^0 \cos(0)$

$\Rightarrow C_1 + 2C_2 = 9 \Rightarrow C_2 = 3$

$\therefore$ the unique solution is $\boxed{y(x) = 3e^x \cos(2x) + 3e^x \sin(2x)}$

(β)

Question 4 (6 marks) Find the general solutions of the differential equations:

(a) $y''' - 7y' + 6y = 0$

the char. eq. is $\lambda^3 - 7\lambda + 6 = 0$, by inspection, $\lambda = 1$ is a root, so $\lambda^3 - 7\lambda + 6 = (\lambda - 1)(\lambda^2 + \lambda - 6)$

$= (\lambda-1)(\lambda+3)(\lambda-2)$

so roots are $\lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = -3$

and the general solution $\boxed{y(x) = c_1 e^{x} + c_2 e^{2x} + c_3 e^{-3x}}$

(b) $x^3y''' + x^2y'' - 2xy' + 2y = 0, \quad x > 0$

the char. eq. is $m(m-1)(m-2) + m(m-1) - 2m + 2 = 0$

or $m(m-1)(m-2) + m(m-1) - 2(m-1) = 0$

so $(m-1)[m(m-2) + m - 2] = 0$

thus $(m-1)(m-2)(m+1) = 0$

so roots are $m_{1,2} = \pm 1,\ m_2 = 2$ and the general solution is $\boxed{y(x) = c_1 x + c_2 x^{-1} + c_3 x^2}$

(B)

Question 5 (6 marks)
(a) Consider $f(x) = x^3 + 8x - 7$. If you were to use Fixed Point Iteration to find the root of this function in the interval $[0,1]$, what would you use for $g(x)$ and why?

$f(x) = 0 \Rightarrow x^3 + 8x - 7 = 0 \Rightarrow 8x = 7 - x^3 \Rightarrow x = \frac{7-x^3}{8}$

so $\boxed{g(x) = \frac{7-x^3}{8}}$ fixed point condition

then $\boxed{|g'(x)| = \frac{3}{8}x^2 < 1 \text{ on } [0,1]}$ convergent sequence

(b) Use Newton's Method to find the root of this $f(x)$ to 4 decimal places. Start with $x_0 = 1$. Verify your answer.

$f(x) = x^3 + 8x - 7 \Rightarrow f'(x) = 3x^2 + 8$

so $x_{n+1} = x_n - \frac{x_n^3 + 8x_n - 7}{3x_n^2 + 8} = \frac{2x_n^3 + 7}{3x_n^2 + 8}$

$x_0 = 1, \quad x_1 = \frac{2+7}{3+8} = \frac{9}{11} = 0.8182$

$x_2 = 0.8089 = x_3$ so stop

$\boxed{f(0.8089) \approx 4.8 \times 10^{-4}}$ so okay $\therefore$ root is $\boxed{0.8089}$

(c) If we were going to find the interpolating polynomial $p_n(x)$ for 5 data points, what is the maximum possible degree that the polynomial could have?

(A) 2 (B) 3 (C) 4 (D) 5 (E) 6 (F) impossible to answer

$(n+1 = 5)$