Engineering Economics
ECO 1192

Topic 2: Time-value mechanics

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Recommended readings

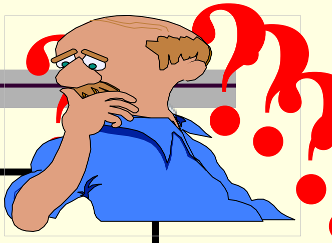
- Fraser et al.** chapters 2 and 3
  - Newnan et al. chapters 3 and 4
  - Park chapters 2 and 3

** Fraser, Jewkes and Pirnia, Engineering Economics, 6th edition, Pearson Education, Toronto, Ontario, 2017.

Lecture objectives

1. Define and calculate
   - Simple and compound interest rates
     - nominal, effective and actual
   - Discrete and continuous cash flows
2. Develop different cash flow patterns
3. Understand interest rate factors
4. Combine interest rates and cash flows
5. Apply summary measures (e.g., Present Worth)

Which projects are valid (acceptable)? Which project (if any) is best?



Project parameters	**Project A**	**Project B**	**Project C**
First cost ($)	**3,000**	**5,000**	**8,000**
Annual cost ($)	**600**	**900**	**1300**
Annual revenues ($)	**1500**	**1750**	**2000**
Salvage value ($)	**0**	**-200**	**1000**
Duration (years)	**5**	**10**	**20**
Interest rate (10%)	**10**	**10**	**10**

What is a valid or acceptable project?

- A project for which cash inflows (revenues if a private project and benefits if a public project) exceed cash outflows (costs) after adjusting for the time at which cash inflows and outflows occur during the life of the project.
  - Based on the assumption that a dollar today is worth more than a dollar tomorrow.

Frequently used symbols

- r ≡ nominal or market interest rate
- i ≡ effective interest rate
- n ≡ number of years
- m ≡ number of within compounding periods
  - if 5% compounded **monthly**, m = 12
- P or PW ≡ Present worth
  - NPW ≡ Net Present Worth
- AEW ≡ Annual Equivalent Worth
- F or FW ≡ Future worth
  - NFW ≡ Net Future Worth

Working assumptions

1. Discrete (not continuous) cash flows and discrete compounding
2. Ownership capital only (no debt capital)
3. No price changes (neither inflation nor deflation during a project's life)
4. Unlimited funds (no capital budgeting): can select all acceptable independent projects
5. Total certainty (no uncertainty or risk)
6. No government (no taxes; no depreciation of capital assets)
7. No imponderables (all project benefits and costs have been converted to their equivalent monetary value)
   - Value of time and lives saved ……………

Time value of money: equivalence

- Time value of money
  - Consumers prefer consuming a good or service NOW rather than later.
  - The stronger the preference for consuming NOW over later, the more important is the required compensation.
  - We will use interest rates to measure the strength of this time-related preference.
- Equivalence
  - Defined as project outflows (i.e., costs) at one point during the life of a project equal to project benefits (revenues) at another time.
  - Equivalence can be defined as
    - Mathematical
    - Decisional
    - Market

Cash flow diagrams

- **Arrows**
  - interpretation depends on one's perspective
    - positive for **cash inflows** (e.g., revenues)
    - negative for **cash outflows** (e.g., expenses)
  - length represents the magnitude ($) of the cash flow
- When cash flows occur must be explicitly stated.
- The interest rate and its frequency of calculation must be known (provided).

Interest Rates

- A measure of the importance of time in economic analysis.
- Rate
  - received by a lender for lending money (i.e., foregoing current consumption for future consumption)
  - paid by a borrower for the use of a lender's funds.

Simple interest (No compounding)

- Interest income is based on
  - the number of finite interest periods per year
  - the amount of money initially borrowed or loaned
- Interest income earned in previous periods has no bearing on the interest income earned this period.

$$F_n = P + Pin = P(1+ in)$$

  - where P = amount invested (borrowed)
  - F = future amount = interest income + P
  - i = the rate of interest earned per period on the amount invested
  - n = the number of interest periods (e.g., number of years)

Compound Interest Rates: Discrete or Continuous

1. Discrete Compounding
   - Simple interest rate (**of no interest in this course**)
     - No within-year compounding (m = 0)
   - Compound interest rate
     - Finite number (m) of within-year compounding periods
     - 12% compounded monthly (m = 12)
     - 12% compounded quarterly (m = 4)
2. Continuous Compounding
   - Infinite number of compounding periods ($m \rightarrow \infty$)
     - 12% compounded continuously

Discrete Interest: compounded

- Interest income is based on
  - number of interest periods
  - interest earned or owed in each interest period
  - the amount of money borrowed or loaned
- $F = P(F/P, i\%, N)$ i.e., future worth
- $I = [P(1+i)^N - P]$ ➔ interest income
  - $F_1 = P + Pi = P(1+i)$ [end of year 1]
  - $F_2 = P(1+i)^i + P(1+i) = P(1+i)^2$ [end of year 2]
  - $F_3 = P(1+i)^2 i + P(1+i)^2 = P(1+i)^3$ [end of year 3]

General Formula: $\mathbf{F_n = P(1+i)^n}$

Please refer to Word document #2 Decision Criteria Interest Factor Derivation in the course website folder "ECO 1192 Documents on Virtual Campus".

Discrete Interest Rate Calculations

- $1,000 borrowed for 3 years at 10% compounded annually.
- What is the amount owed at the end of three years if the money is borrowed at
  - simple interest of 10% per year? [Example 1]
  - 1% compounded annually? [Example 2]

Simple vs Compound Interest

Example 1: Simple Interest

End of year (EOY)	Amount Borrowed	Interest for Period (e.g., year)	Amount owed at EOY
0	$1,000		
1		$100	$1,100
2		100	1,200
3		100	1,300

Example 2: Compound Interest

EOY	Amount Borrowed	Annual Interest	Balance at EOY
0	$1,000		
1		$100	$1,100
2		110	1,210
3		121	1,331

Interest Rate Compounding

- What is the frequency of within-year compounding?
  - How many times is interest income (or expense) calculated each year?
- Discrete interest rates
  - Nominal
  - Effective
  - Actual interest rate
- Continuous interest rates
  - Nominal
  - Effective
  - Actual interest rate ≈ 0 since $m \rightarrow \infty$

Nominal Interest Rates

- The complete description of an interest rate has a quantitative and a qualitative component.
  - Quantitative compounded = interest rate percentage only (e.g., 15%)
  - Qualitative = the within-year frequency of compounding (i.e., once, twice … per year)
- The nominal rate captures the **quantitative dimension ONLY** of an interest rate
  - no reference to the within-year frequency of compounding
  - If the interest rate is defined as
    - 15% compounded annually, the nominal rate is 15%
    - 12% compounded semi-annually, the nominal rate is 12%
    - 10% compounded monthly, the nominal rate is 10%
    - 15% compounded daily, the nominal rate is 15%
- The nominal rate is usually stated on an annual basis.

Discrete Effective and Actual Interest Rates

Effective Interest Rate (i)

- Combines the quantitative and qualitative components of an interest rate (e.g., 15% compounded semi-annually)
- converts an interest rate with one or more within-year compounding periods to an equivalent interest rate compounded annually

  $= (F - P)/P = \{P(1+r/m)^m - P\}/P$ [applies to one year only]

  $= \mathbf{(1+r/m)^m - 1}$
- Usually stated on an annual basis

Actual Interest Rate (= r/m)

- the nominal rate (r) per within-year compounding periods (m)
  - 12% compounded semi-annually ➔ 6%
  - 12% compounded monthly ➔ 1%

Example: Discrete Interest Rate

- Given an interest rate of 12% compounded quarterly
  - Nominal rate = 12%
  - Effective rate = $[1+(0.12/4)]^4 - 1 = 0.1255$ = 12.55%
  - Actual rate = 12%/4 = 3% per quarter
    - Investing $1 at <u>3%</u> per quarter is **equivalent** to investing $1 at <u>12.55%</u> annually
    - If you invest one dollar, your account balance after 1 year would be $1.13 in both cases.

Excel Function: EFFECT (r,m)

- Calculates the effective interest rate that corresponds to a nominal rate (r) with “m” within-year compounding periods.

<u>Example</u>

Given a nominal rate of 10%

- compounded semi-annually

  =EFFECT(10%,2) = 0.1025 or 10.25%

- compounded quarterly

  =EFFECT(10%,4) = 0.103813 or 10.38%

- compounded monthly

Please refer to PowerPoint document #4 <u>Key Excel Functions in Engineering Economics</u> in the course website folder “ECO 1192 Documents on Virtual Campus”.

Function names are not case sensitive.

Financial Table: Discrete Cash Flows and Discrete Compounding

		DISCRETE CASH FLOW and DISCRETE COMPOUNDING (10%)					
n	**(F/P,i%,n)**	**(P/F,i%,n)**	**(A/P,i%,n)**	**(P/A,i%,n)**	**(A/F,i%,n)**	**(F/A,i%,n)**	**(A/G,i%,n)**
1	**1.1000**	**0.9091**	**1.1000**	**0.9091**	**1.0000**	**1.0000**	**0.0000**
2	**1.2100**	**0.8264**	**0.5762**	**1.7355**	**0.4762**	**2.1000**	**0.4762**
3	**1.3310**	**0.7513**	**0.4021**	**2.4869**	**0.3021**	**3.3100**	**0.9366**
4	**1.4641**	**0.6830**	**0.3155**	**3.1699**	**0.2155**	**4.6410**	**1.3812**
5	**1.6105**	**0.6209**	**0.2638**	**3.7908**	**0.1638**	**6.1051**	**1.8101**

Continuous Nominal Interest Rates (r)

- The complete description of an interest rate has a quantitative and a qualitative component.
  - Quantitative compounded = interest rate percentage (i.e., 15%)
  - Qualitative = the within-year frequency of compounding (i.e., once, twice … per year)
- Captures the **quantitative dimension ONLY** of an interest rate
  - no reference to the within-year frequency of compounding
  - If the interest rate is defined as
    - 15% compounded continuously ➔ nominal rate = 15%
    - 10% compounded continuously ➔ nominal rate = 10%
- The nominal rate is usually stated on an annual basis

Continuous Effective and Actual Interest Rates

<u>Effective Interest Rate (i)</u>

- Combines the quantitative and qualitative components of an interest rate (e.g., 15% compounded continuously)
- Converts an interest rate with an infinite number of within-year compounding periods to an equivalent interest rate compounded annually
  - Example: 10% compounded continuously
- Modify the discrete effective rate = $(1+r/m)^m - 1$ as follows:

$$\{[1+1/(m/r)]^{m/r}\}^r - 1$$

- As "m" ➔ ∞, the effective annual interest rate tends to

$$(e^r - 1)$$

- Usually stated on an annual basis

<u>Actual Interest Rate (r/m)</u>

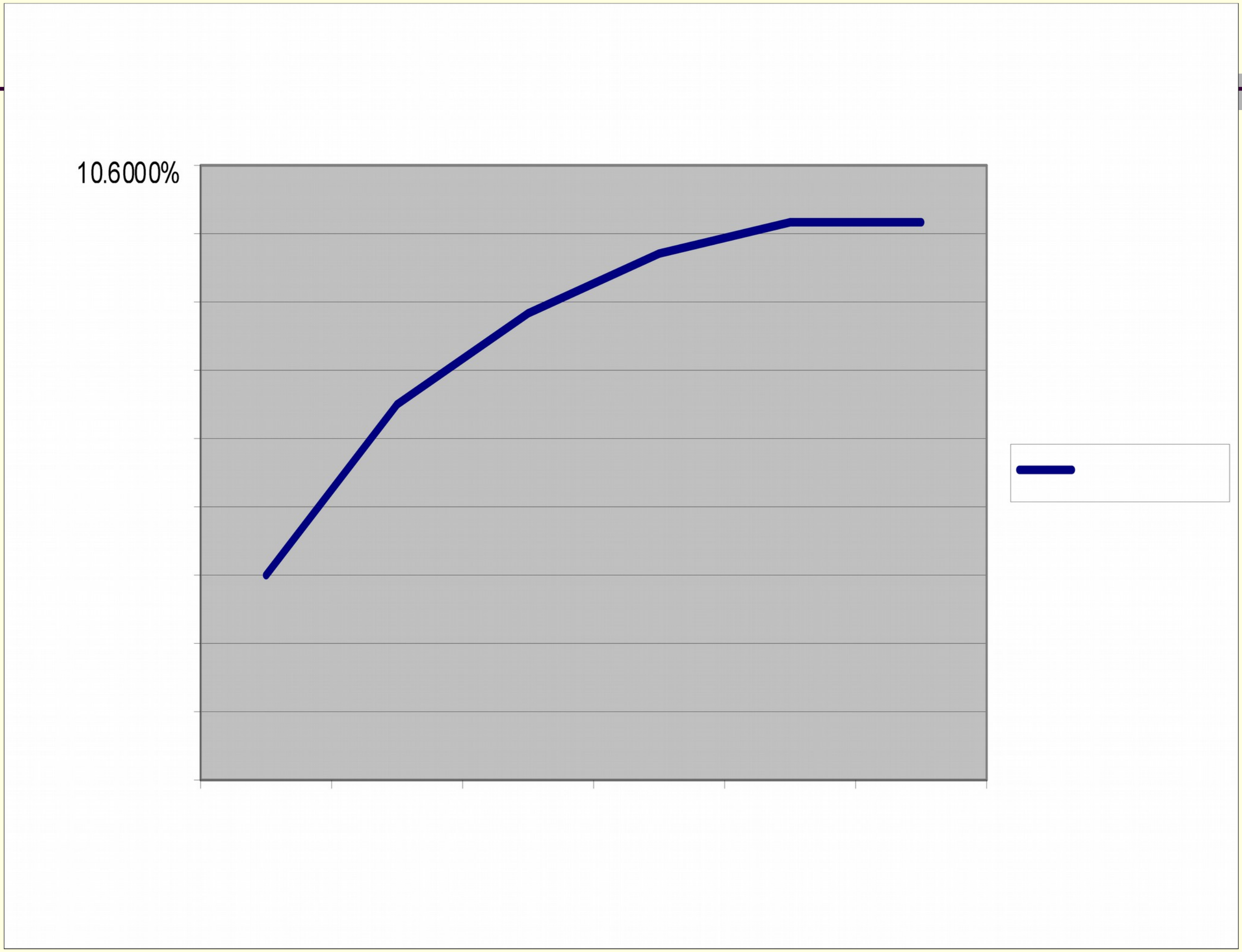
- actual interest rate $(r/m) \approx 0\%$

Example: Continuous Interest Rate

Given: Interest rate of 15% compounded continuously

1. Nominal rate = 15%
2. Effective rate: **($e^{0.15 - 1}$) = 0.1618 or 16.18% compounded <u>annually</u>**
3. Actual rate ➔ (0.15/∞) ➔ 0%

Nominal Interest Rate = 10%



- The effective rate increases as the within-year frequency of compounding (m) increases.
- For example, the effective rate for monthly compounding is larger than for quarterly compounding.
- The maximum rate occurs when within-year compounding is infinite.

Interest Rate and Cash Flow Scenarios

Note: In all calculations, the frequency of within-year compounding MUST match the cash flow frequency (e.g., monthly cash flows AND monthly compounding).

	Compound Interest Rate	
	1. DISCRETE RATE	2. CONTINUOUS RATE
Cash Flows ↓		
1. DISCRETE	A**	B
2. CONTINUOUS	C	D

** Major focus of problems in this course

Scenario A**

Discrete Compounding and Discrete Cash Flows

- Most common situation (Mismatch)
  - cash flow frequency and interest compounding differ
    - 10 **annual** deposits in a bank account that pays interest at 5% **compounded quarterly**
- Three possible situations

1. frequency of within-year compounding = cash flow frequency
2. frequency of within-year compounding > cash flow frequency
3. frequency of within-year compounding < cash flow frequency

CASH FLOWS ↓	INTEREST RATE COMPOUNDING SITUATIONS: 1. DISCRETE RATE	INTEREST RATE COMPOUNDING SITUATIONS: 2. CONTINUOUS RATE
1.DISCRETE	A**	B
2.CONTINUOUS	C	D

Interest Rate and Cash Flow Mismatch

- Formula for determining the effective interest rate that corresponds to the frequency of the cash flow. The interest rate per cash flow period is

$$(1+r/m)^{(m/k)} - 1$$

where

  - "m" is the number of within-year interest periods
    - if quarterly compounding, m=4
  - "r" is the nominal rate of interest per year
  - "k" is the frequency of cash flow per year

Example
Frequency Mismatch

- Quarterly deposits of $1,000
- Interest rate of 12% compounded monthly
- The first $1,000 deposit is made in 3 months from now.
- **Find the dollar value of the deposits after 5 years?**

- Using frequency conversion formula with m = 12 and k = 4, the effective quarterly interest rate becomes $(1+[0.12/12])^{(12/4)} - 1 = 0.0303$ or 3.03% per quarter
- F = A(F/A,i%,N)
- F = 1,000[F/A,3.03%,5(4)]
  F = 1,000(F/A,3.03%,20) = 1,000(26.9525)
  F = $26,952.50
- The “F = A(F/A,i%,N)” formula applies only to discrete interest and discrete cash flow situations.

Scenario B
Continuous Compounding: Discrete Cash Flows

- Interest credited (or calculated) on an on-going basis [e.g., 5% compounded continuously]
- Deposits (or payments) are made at fixed times during a year (e.g., $200 at the end of each 5-year period)
- Example: How much money would you have in an account fifteen (15) years from now if you made fifteen (15) end-of-year deposits of $1,000 each?
  - The first $1,000 deposit occurs one year from now and the account pays interest at a 10% rate compounded continuously.

CASH FLOWS ↓	INTEREST RATE COMPOUNDING SITUATIONS	
	1. DISCRETE RATE	2. CONTINUOUS RATE
1.DISCRETE	A	B
2.CONTINUOUS	C	D

Scenario B

Continuous Compounding & Discrete Cash Flows

There are two possible solutions to the previous example

1. Use the appropriate interest rate factor for calculating the Future Worth of discrete cash flow and continuous interest compounding situations
   - discussed later
2. Convert the 10% continuous interest rate to an equivalent annual compounding interest rate and use the appropriate discrete cash flow and discrete compounding interest factor

CASH FLOWS ↓	INTEREST RATE COMPOUNDING SITUATIONS	
	1. DISCRETE RATE	2. CONTINUOUS RATE
1.DISCRETE	A	**B**
2.CONTINUOUS	C	D

Continuous Cash Flows

1. **uniform (step)**
   - Continuous, constant cash flow throughout a period of time (e.g., one year) which is equivalent to an aggregate amount
2. gradient (ramp)
3. decay
4. exponential
5. Growth

** Please note that continuous cash flow patterns 2, 3, 4 and 5 are not required for this course.

SCENARIO C

Discrete Compounding & Continuous Cash Flows

- The cash flow frequency has no effect on the interest owed or owing during specific periods of the year.

<u>Example</u>

- Busy toll booth on a major New England highway collects about $1 million in fees evenly distributed throughout the year (day and night).
- Interest rate is 12% compounded quarterly
- Actual interest rate = 3% per quarter
- Effective cash flow = $1M/4 or $250K per quarter.

CASH FLOWS ↓	INTEREST RATE COMPOUNDING SITUATIONS	
	1. DISCRETE RATE	2. CONTINUOUS RATE
1.DISCRETE	A	B
2.CONTINUOUS	C	D

SCENARIO C

Discrete Compounding & Continuous Cash Flows

- No requirement for a "special" interest factor since the continuous cash flow can be partitioned (divided) to match the frequency of interest rate compounding.
- In the previous example, the $1,000 (over one year) continuous cash flow is equivalent to a **quarterly** cash flow of $250 to match the 3% interest rate **per quarter.**

CASH FLOWS ↓	INTEREST RATE COMPOUNDING SITUATIONS	
	1. DISCRETE RATE	2. CONTINUOUS RATE
1.DISCRETE	A	B
2.CONTINUOUS	C	D

SCENARIO D
Continuous Compounding & Cash Flow

Example

Constant cash flows equivalent to $500 annually are deposited in a bank account that pays 5% interest compounded continuously.

- Special interest factors are needed in continuous cash flow and compounding situations

CASH FLOWS ↓	INTEREST RATE COMPOUNDING SITUATIONS	
	1. DISCRETE RATE	2. CONTINUOUS RATE
1.DISCRETE	A	B
2.CONTINUOUS	C	**D**

Interest Rate Factors

- Algebraic short forms (or shortcuts) for dealing with frequently encountered cash flow and compounding situations
- Example (F/P,i%,N)
  - Find the future worth (F) of a present amount (P) given an interest rate of "i%" and the number of interest periods "N" (if annual compounding).
- Interest factors are used for the following types of situations:
  - single sums (mostly present and future values)
  - uniform or constant series
  - gradient (arithmetic and geometric) series

Formulas Provided at All Examinations

	Cash Flow	Discrete	Discrete	Continuous
	Compounding	Discrete	Continuous	Continuous
SINGLE SUMS	*Compound Amount*	$F = P(1+i)^n = P(F/P,i,n)$	$F=Pe^{rn} = P(F/P,r,n)$	$F=Pe^{rn} = P(F/P,r,n)$
	Discount Amount	$P = F(1+i)^{-n} = F(P/F,i,n)$	$P=Fe^{-rn} = F(P/F,r,n)$	$P=Fe^{-rn} = F(P/F,i,n)$
	Compound Amount	$F = A[\frac{(1+i)^n-1}{i}]$ $F = A(F/A,r,n)$	$F = A[\frac{e^{rn}-1}{e^r-1}]$	$F = \bar{A}[\frac{e^{rn}-1}{r}]$
	Sinking Fund	$A = F[\frac{i}{(1+i)^n-1}]$ $A = F(A/F,r,n)$	$A = F[\frac{e^r-1}{e^{rn}-1}]$	$\bar{A} = F[\frac{r}{e^{rn}-1}]$
	Discount Amount	$P = A[\frac{(1+i)^n-1}{i(1+i)^n}]$ $P = A(P/A,r,n)$	$P = A[\frac{1-e^{-rn}}{e^r-1}]$	$P = \bar{A}[\frac{e^{rn}-1}{re^{rn}}]$
	Capital Recovery	$A = P[\frac{i(1+i)^n}{(1+i)^n-1}]$ $A = P(A/P,r,n)$	$A = P[\frac{e^r-1}{1-e^{-rn}}]$	$\bar{A} = P[\frac{re^{rn}}{e^{rn}-1}]$

Formulas Provided at All Examinations

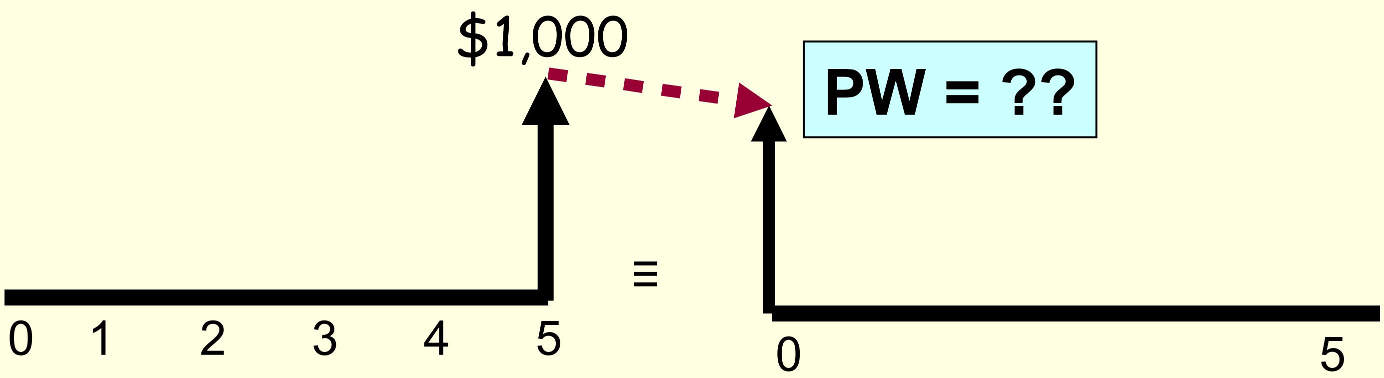
Cash Flow	Discrete	Discrete	Continuous
Compounding	Discrete	Continuous	Continuous
Uniform Gradient Series (Conversion to a Uniform Series)	$A = G[\frac{1}{i} - \frac{n}{(1+i)^n - 1}]$ A = G(A/G,r,n)	$A = G[\frac{1}{e^r - 1} - \frac{n}{e^{rn} - 1}]$	Not defined.
Geometric Gradient (Conversion to a Single Sum i.e. Present Worth)	***If i% ≠ k%:*** P = C(P/C,i,k,N) $P = [\frac{C}{(i-k)}](1 - [\frac{(1+k)}{(1+i)}]^n)$ ***If i% = k%*** $P = \frac{CN}{(1+i)}$ C ≡ First (≠0) term of series	Not defined	Not defined

Discrete Single Sums: Present (PW) and Future Worth (FW)

INTEREST FACTOR	**NOTATIONAL FORM**	**ALGEBRAIC EQUIVALENT**
1. Compound amount factor	**(F/P,i%,n) [Find F given P, i% and n]**	**$F = P(1+i)^n$**
2. Discount amount factor	**(P/F,i%,n) [Find P given F, i% and n]**	**$P = F(1+i)^{-n}$**

Cash Flow Diagram: Single Sum (PW)

PW = 1,000(P/F,10%,5)



What is the value TODAY of $1,000 in five years if the rate of interest is 10% compounded annually?

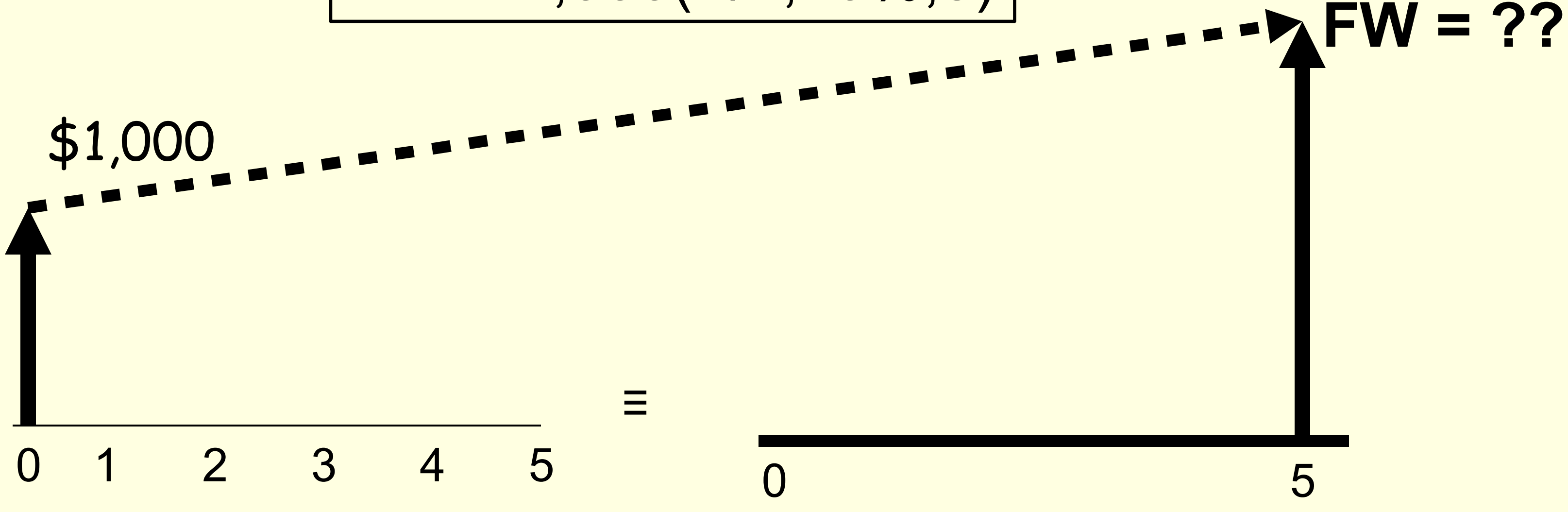
Financial Table

		DISCRETE CASH FLOW AND COMPOUNDING (10%)					
n	(F/P,i%,n)	(P/F,i%,n)	(A/P,i%,n)	(P/A,i%,n)	(A/F,i%,n)	(F/A,i%,n)	(A/G,i%,n)
1	1.1000	0.9091	1.1000	0.9091	1.0000	1.0000	0.0000
2	1.2100	0.8264	0.5762	1.7355	0.4762	2.1000	0.4762
3	1.3310	0.7513	0.4021	2.4869	0.3021	3.3100	0.9366
4	1.4641	0.6830	0.3155	3.1699	0.2155	4.6410	1.3812
5	1.6105	0.6209	0.2638	3.7908	0.1638	6.1051	1.8101

PW = F(P/F,i%,N) = 1,000(P/F,10%,5) = 1,000(0.6209) = $620.09

Cash Flow Diagram: Single Sum (FW)

FW = 1,000(F/P,10%,5)



How much is $1,000 TODAY worth in five years if the rate of interest is 10% compounded annually?

Financial Table

		DISCRETE CASH FLOW AND COMPOUNDING (10%)					
n	(F/P,i%,n)	(P/F,i%,n)	(A/P,i%,n)	(P/A,i%,n)	(A/F,i%,n)	(F/A,i%,n)	(A/G,i%,n)
1	1.1000	0.9091	1.1000	0.9091	1.0000	1.0000	0.0000
2	1.2100	0.8264	0.5762	1.7355	0.4762	2.1000	0.4762
3	1.3310	0.7513	0.4021	2.4869	0.3021	3.3100	0.9366
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5	1.6105	0.6209	0.2638	3.7908	0.1638	6.1051	1.8101

FW = P(F/P,i%,N) = 1,000(F/P,10%,5) = 1,000(1.6105) = $1,610.05

Examples: Single Sums

Example

What is the value 5 years from now (future value) of $100 invested at 10% per year?

$$\mathbf{FW = P(1+i)^N = 100(1+0.10)^5 = \$161.05}$$

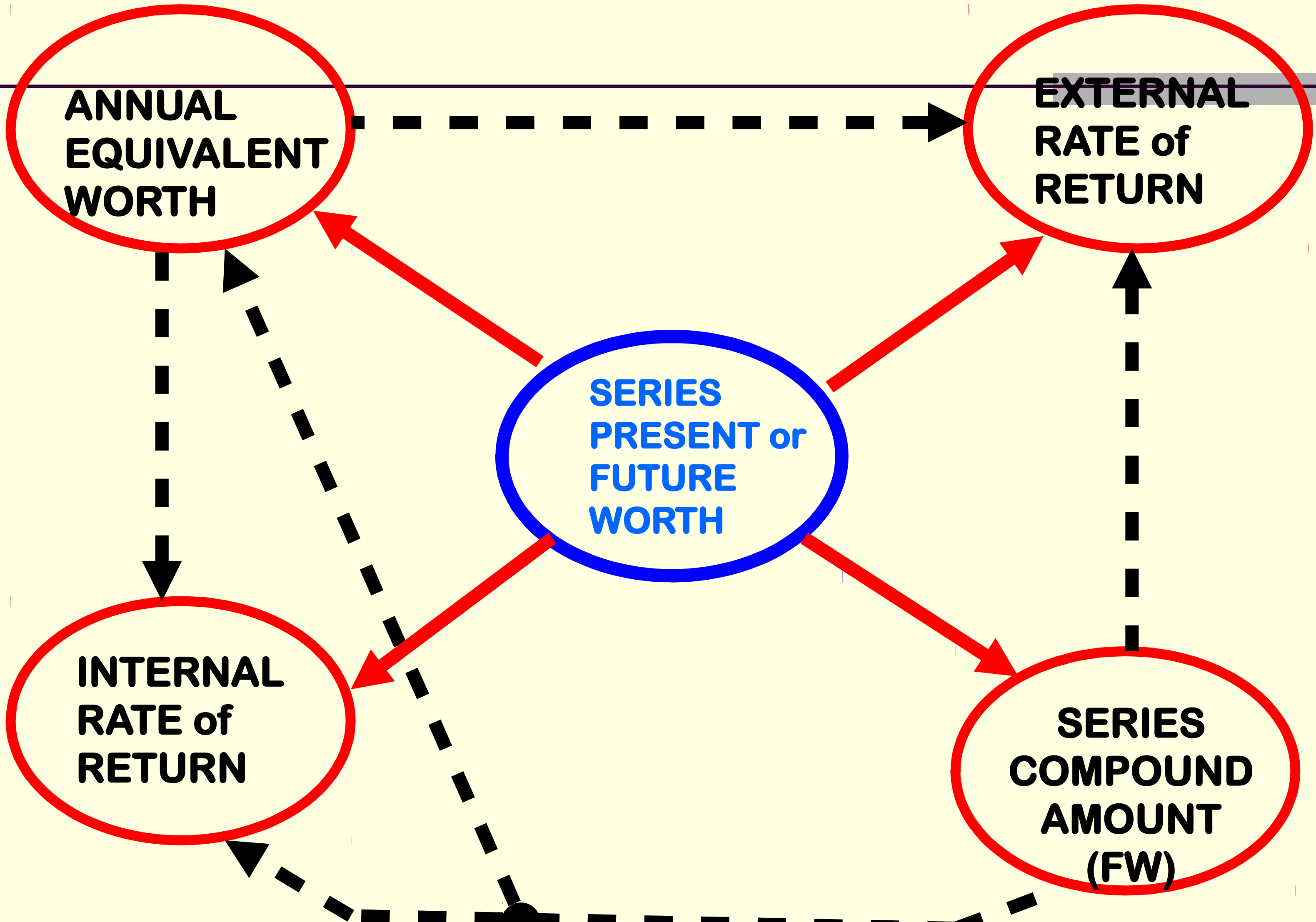
Example

What is the present value of $1,000 in 20 years if the interest rate is 6% compounded annually?

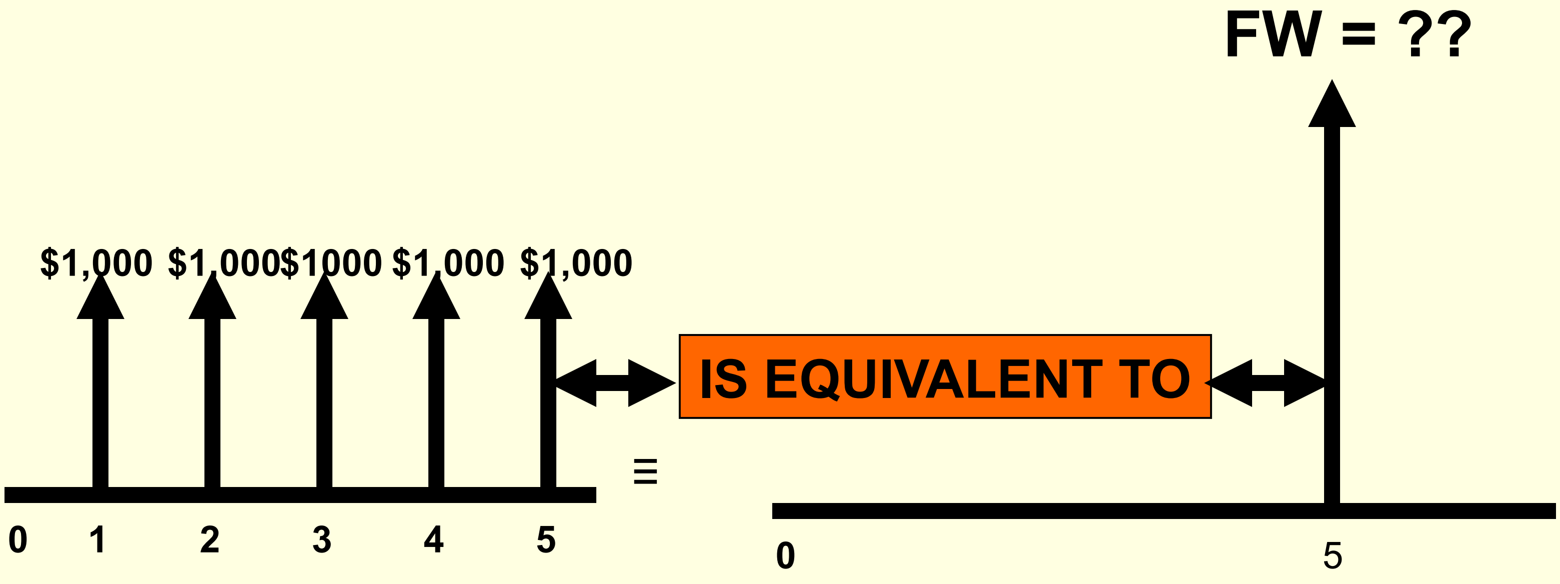
$$\mathbf{PW = F(1+i)^{-N} = F(1+0.06)^{-20} = \$311.80}$$

You would be indifferent between $311.80 now and $1,000 in 20 years. The difference between $1,000 and $311.80 is interest income earned during the 20 years.

Conversion of Series Present Worth



Annuities



Find the value in five years of annual deposits of $1,000 over five years beginning in one year from today?

Financial Table

DISCRETE CASH FLOW AND COMPOUNDING (10%)							
n	(F/P,i%,n)	(P/F,i%,n)	(A/P,i%,n)	(P/A,i%,n)	(A/F,i%,n)	(F/A,i%,n)	(A/G,i%,n)
1	1.1000	0.9091	1.1000	0.9091	1.0000	1.0000	0.0000
2	1.2100	0.8264	0.5762	1.7355	0.4762	2.1000	0.4762
3	1.3310	0.7513	0.4021	2.4869	0.3021	3.3100	0.9366
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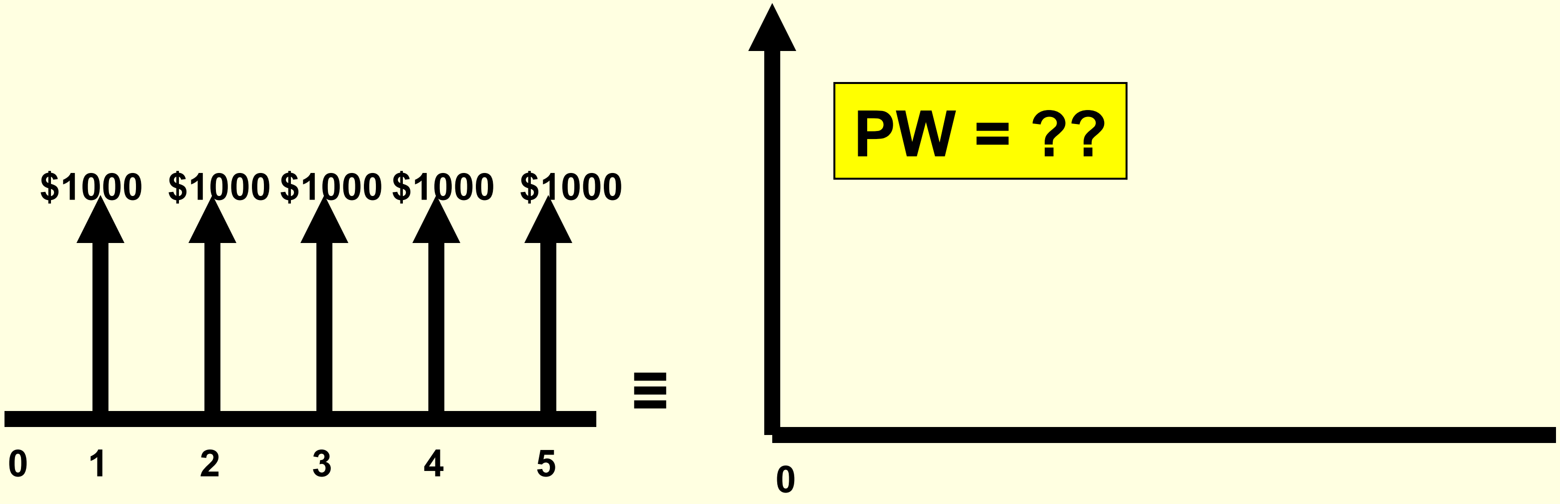
5	1.6105	0.6209	0.2638	3.7908	0.1638	6.1051	1.8101

F = P(F/A,i%,N) = 1,000(F/A,10%,5) = 1,000(6.1051) = $6,105.10

Annuities



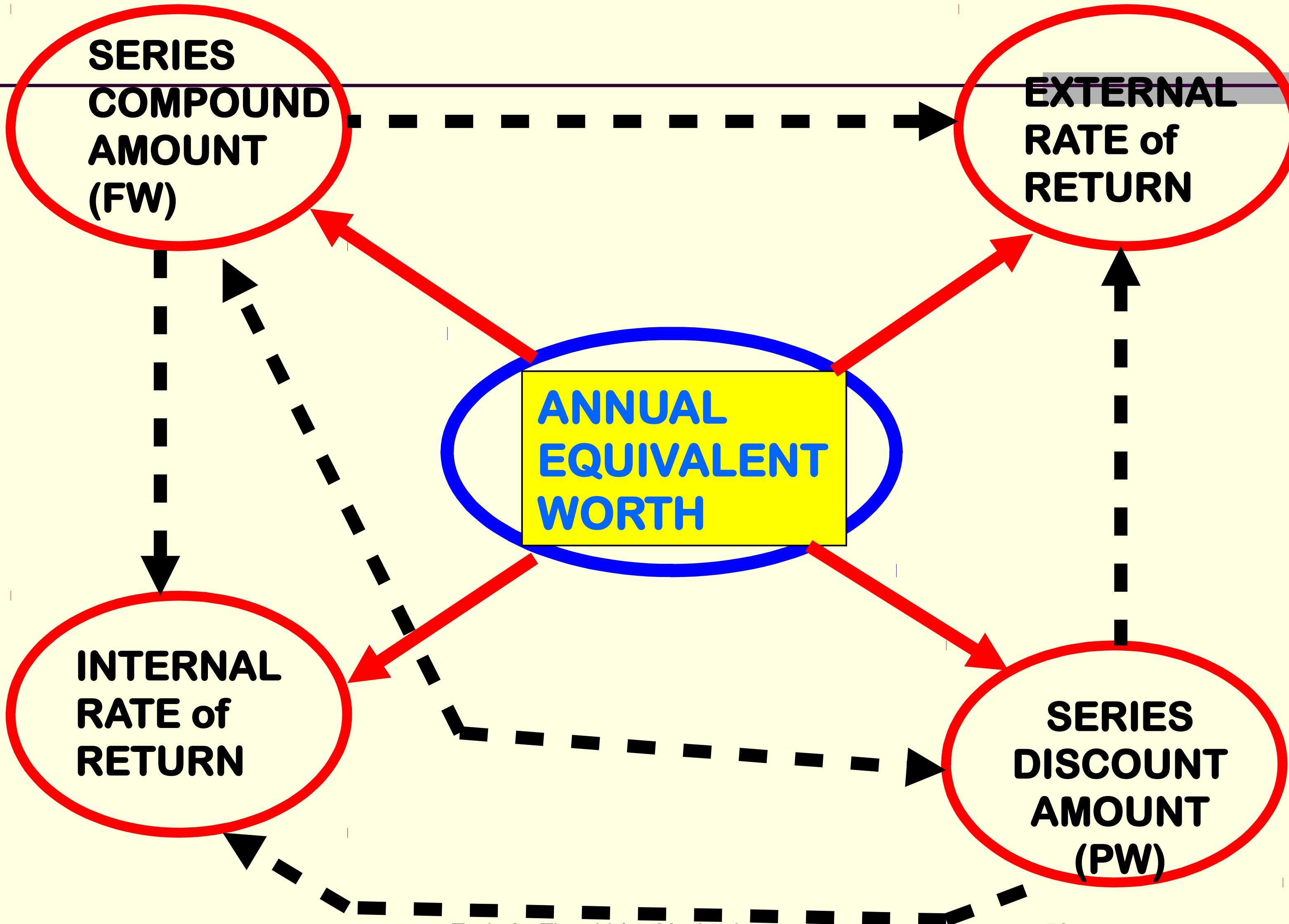
Find the Present Worth of five annual deposits of $1,000 beginning in one year from now?

Financial Table

		DISCRETE CASH FLOW AND COMPOUNDING (10%)					
n	(F/P,i%,n)	(P/F,i%,n)	(A/P,i%,n)	(P/A,i%,n)	(A/F,i%,n)	(F/A,i%,n)	(A/G,i%,n)
1	1.1000	0.9091	1.1000	0.9091	1.0000	1.0000	0.0000
2	1.2100	0.8264	0.5762	1.7355	0.4762	2.1000	0.4762
3	1.3310	0.7513	0.4021	2.4869	0.3021	3.3100	0.9366
4	1.4641	0.6830	0.3155	3.1699	0.2155	4.6410	1.3812
5	1.6105	0.6209	0.2638	3.7908	0.1638	6.1051	1.8101

P = A(P/A,i%,N) = 1,000(P/A,10%,5) = 1,000(3.7908) = $3,790.80

Conversion of Annual Equivalent Worth (Annuity)



Uniform (Constant or regular)) Series

INTEREST FACTOR	NOTATIONAL FORM	ALGEBRAIC EQUIVALENT
Uniform series compound factor	(F/A,i%,n)	$\{(1+i)^n - 1\}/i$
Uniform series discount factor	(P/A,i%,n)	$\{(1+i)^n - 1\}/\{i(1+i)^n\}$
Sinking fund factor	(A/F,i%,n)	$i/\{(1+i)^n - 1\}$
Capital recovery factor	(A/P,i%,n)	$\{i(1+i)^n\}/\{(1+i)^n - 1\}$

Please refer to PowerPoint document #4 <u>Key Excel Functions in Engineering Economics</u> in the course website folder "ECO 1192 Documents on Virtual Campus".

Example of a uniform or constant series

$800 is deposited annually in a fund for 10 years. If the fund earns interest at the rate of 10% per year, what will be the value of the fund at the end of the tenth year?

FW = A(F/A,i%,N)
= 800(F/A,10%,10)
= 800(15.9374)
= $12,750

		DISCRETE CASH FLOW AND COMPOUNDING					
		10.0%					
n	(F/P,i%,n)	(P/F,i%,n)	(A/P,i%,n)	(P/A,i%,n)	(A/F,i%,n)	(F/A,i%,n)	(A/G,i%,n)
1	1.1000	0.9091	1.1000	0.9091	1.0000	1.0000	0.0000
2	1.2100	0.8264	0.5762	1.7355	0.4762	2.1000	0.4762
3	1.3310	0.7513	0.4021	2.4869	0.3021	3.3100	0.9366
4	1.4641	0.6830	0.3155	3.1699	0.2155	4.6410	1.3812
5	1.6105	0.6209	0.2638	3.7908	0.1638	6.1051	1.8101
6	1.7716	0.5645	0.2296	4.3553	0.1296	7.7156	2.2236
7	1.9487	0.5132	0.2054	4.8684	0.1054	9.4872	2.6216
8	2.1436	0.4665	0.1874	5.3349	0.0874	11.4359	3.0045
9	2.3579	0.4241	0.1736	5.7590	0.0736	13.5795	3.3724
10	2.5937	0.3855	0.1627	6.1446	0.0627	**15.9374**	3.7255
11	2.8531	0.3505	0.1540	6.4951	0.0540	18.5312	4.0641
12	3.1384	0.3186	0.1468	6.8137	0.0468	21.3843	4.3884
13	3.4523	0.2897	0.1408	7.1034	0.0408	24.5227	4.6988
14	3.7975	0.2633	0.1357	7.3667	0.0357	27.9750	4.9955
15	4.1772	0.2394	0.1315	7.6061	0.0315	31.7725	5.2789
16	4.5950	0.2176	0.1278	7.8237	0.0278	35.9497	5.5493
17	5.0545	0.1978	0.1247	8.0216	0.0247	40.5447	5.8071

Example of a uniform or constant series

How much should you spend (i.e., the purchase price) for an energy-saving device with a five-year life (and no salvage value) if the device is to provide annual savings of $200?

Assume end-of-year savings and an interest rate of 10%.

P = A(P/A,i%,N) = 200(P/A,10%,5)

= 200(3.7908) = $758.16

		DISCRETE CASH FLOW AND COMPOUNDING					
		10,00	% DISCRETE RATE OF INTEREST				
n	(F/P,i%,n)	(P/F,i%,n)	(A/P,i%,n)	(P/A,i%,n)	(A/F,i%,n)	(F/A,i%,n)	(A/G,i%,n)
1	1,1000	0,9091	1,1000	0,9091	1,0000	1,0000	0,0000
2	1,2100	0,8264	0,5762	1,7355	0,4762	2,1000	0,4762
3	1,3310	0,7513	0,4021	2,4869	0,3021	3,3100	0,9366
4	1,4641	0,6830	0,3155	3,1699	0,2155	4,6410	1,3812
5	1,6105	0,6209	0,2638	3,7908	0,1638	6,1051	1,8101
6	1,7716	0,5645	0,2296	4,3553	0,1296	7,7156	2,2236
7	1,9487	0,5132	0,2054	4,8684	0,1054	9,4872	2,6216
8	2,1436	0,4665	0,1874	5,3349	0,0874	11,4359	3,0045
9	2,3579	0,4241	0,1736	5,7590	0,0736	13,5795	3,3724
10	2,5937	0,3855	0,1627	6,1446	0,0627	15,9374	3,7255
11	2,8531	0,3505	0,1540	6,4951	0,0540	18,5312	4,0641
12	3,1384	0,3186	0,1468	6,8137	0,0468	21,3843	4,3884
13	3,4523	0,2897	0,1408	7,1034	0,0408	24,5227	4,6988
14	3,7975	0,2633	0,1357	7,3667	0,0357	27,9750	4,9955
15	4,1772	0,2394	0,1315	7,6061	0,0315	31,7725	5,2789
16	4,5950	0,2176	0,1278	7,8237	0,0278	35,9497	5,5493
17	5,0545	0,1978	0,1247	8,0216	0,0247	40,5447	5,8071

Example of a uniform or constant series

Your plans are to purchase a $30,000 sports car when you graduate in four years. How much should you deposit in your bank account at the end of each of the 4 years in order to pay cash for the car? Assume that your deposits earn 10% compounded annually.

A = F(A/F,i%,N)
= 30,000(A/F,10%,4)
= 30,000(0.2155) = $6,465

DISCRETE CASH FLOW AND COMPOUNDING
10.00 % DISCRETE RATE OF INTEREST

n	(F/P,i%,n)	(P/F,i%,n)	(A/P,i%,n)	(P/A,i%,n)	(A/F,i%,n)	(F/A,i%,n)	(A/G,i%,n)
1	1.1000	0.9091	1.1000	0.9091	1.0000	1.0000	0.0000
2	1.2100	0.8264	0.5762	1.7355	0.4762	2.1000	0.4762
3	1.3310	0.7513	0.4021	2.4869	0.3021	3.3100	0.9366
4	1.4641	0.6830	0.3155	3.1699	0.2155	4.6410	1.3812
5	1.6105	0.6209	0.2638	3.7908	0.1638	6.1051	1.8101
6	1.7716	0.5645	0.2296	4.3553	0.1296	7.7156	2.2236
7	1.9487	0.5132	0.2054	4.8684	0.1054	9.4872	2.6216
8	2.1436	0.4665	0.1874	5.3349	0.0874	11.4359	3.0045
9	2.3579	0.4241	0.1736	5.7590	0.0736	13.5795	3.3724
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17	5.0545	0.1978	0.1247	8.0216	0.0247	40.5447	5.8071

Example of a uniform or constant series

A young couple paid $40,000 for expensive household furniture. After a rocky five-year relationship, the couple decided to split.
the furniture was practically worthless after 5 years
the interest rate was 10% compounded annually

What was the annual equivalent value of its furniture?
A = P(A/P,i%,N)
= 40,000(A/P,10%,5)
= 40,000(0.2638) = $10,552

DISCRETE CASH FLOW AND COMPOUNDING
10.00 % DISCRETE RATE OF INTEREST

n	(F/P,i%,n)	(P/F,i%,n)	(A/P,i%,n)	(P/A,i%,n)	(A/F,i%,n)	(F/A,i%,n)	(A/G,i%,n)
1	1.1000	0.9091	1.1000	0.9091	1.0000	1.0000	0.0000
2	1.2100	0.8264	0.5762	1.7355	0.4762	2.1000	0.4762
3	1.3310	0.7513	0.4021	2.4869	0.3021	3.3100	0.9366
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6	1.7716	0.5645	0.2296	4.3553	0.1296	7.7156	2.2236
7	1.9487	0.5132	0.2054	4.8684	0.1054	9.4872	2.6216
8	2.1436	0.4665	0.1874	5.3349	0.0874	11.4359	3.0045
9	2.3579	0.4241	0.1736	5.7590	0.0736	13.5795	3.3724
10	2.5937	0.3855	0.1627	6.1446	0.0627	15.9374	3.7255
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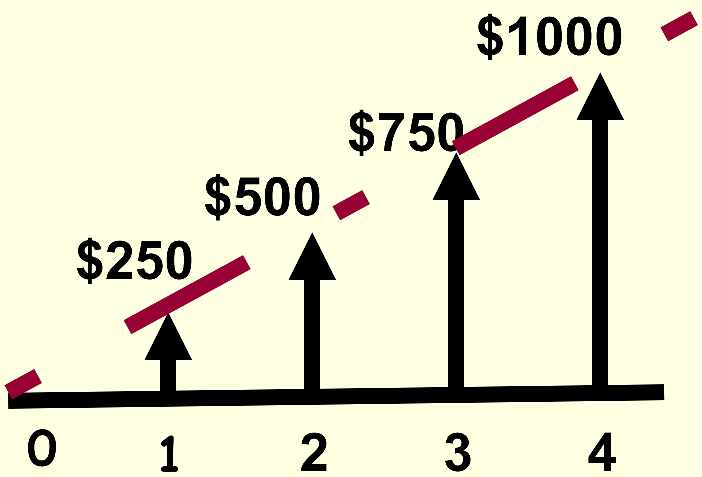
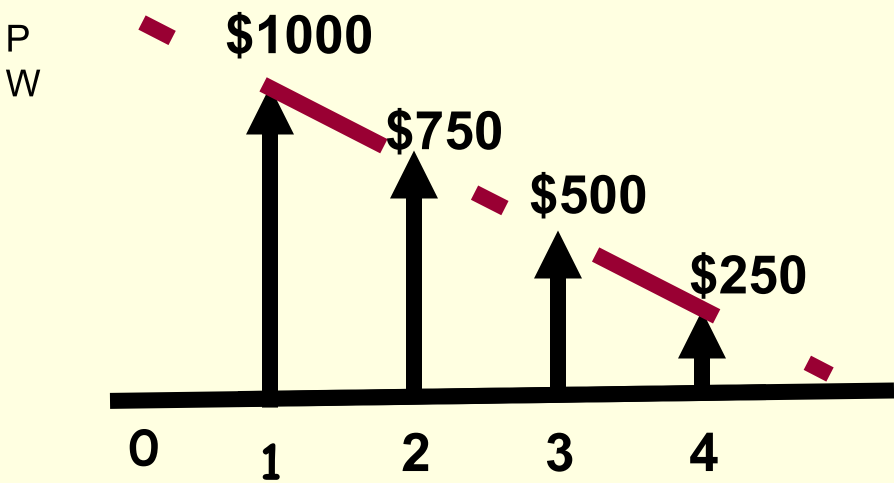
17	5.0545	0.1978	0.1247	8.0216	0.0247	40.5447	5.8071

Linear Gradient Series (Arithmetic Series)





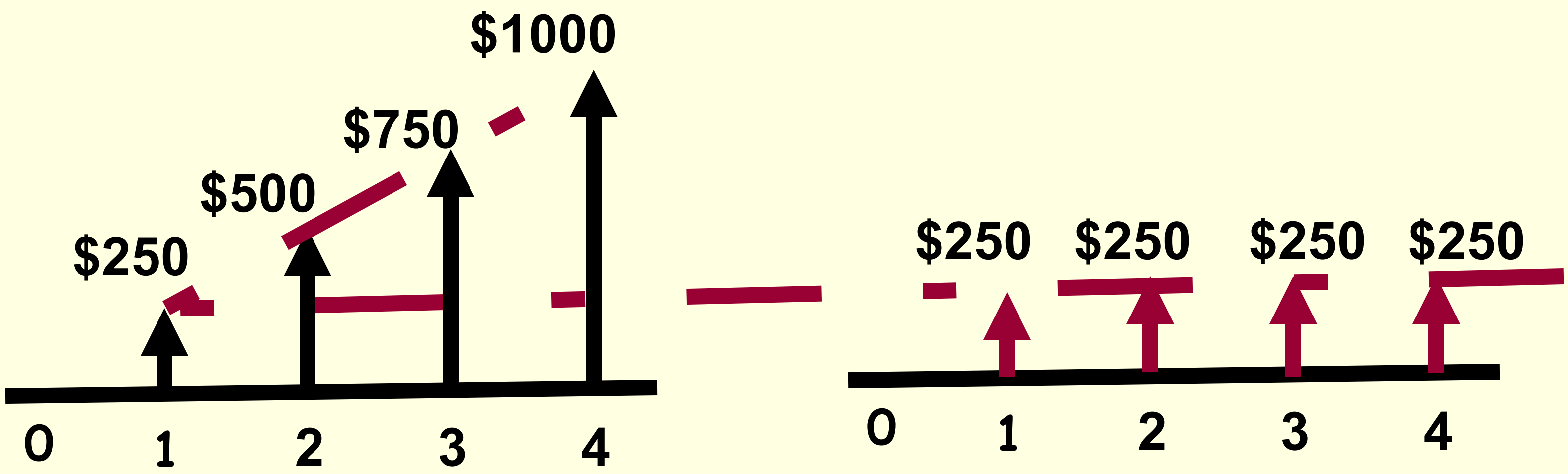
Increasing or growing series

Decreasing or decaying series

Conversion of linear gradient series to an annuity (uniform series)

Original Series

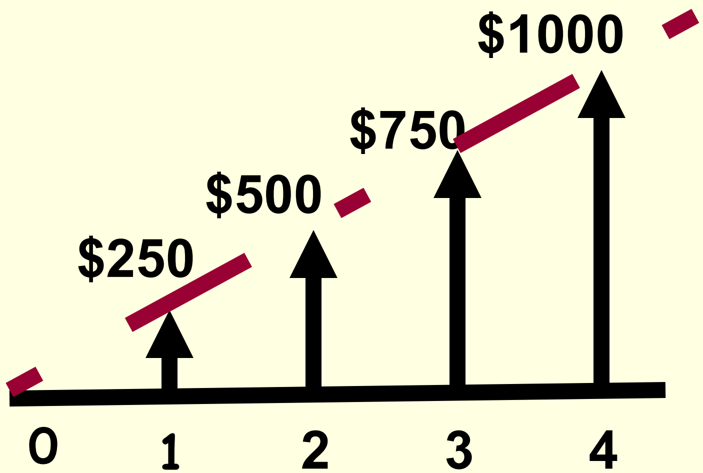
Step 1: Duplicate first element throughout series.



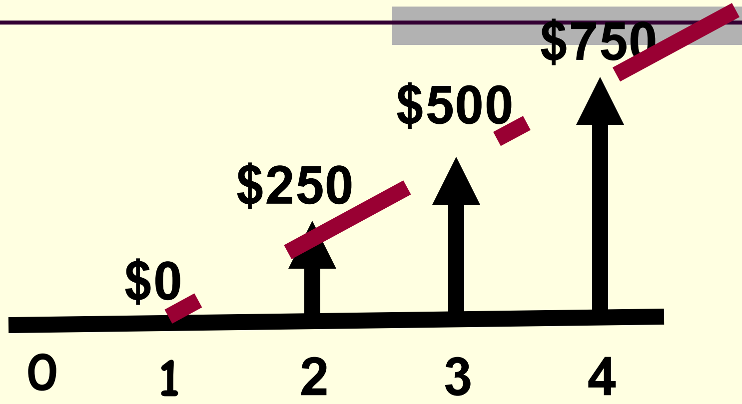
Increasing or growing series

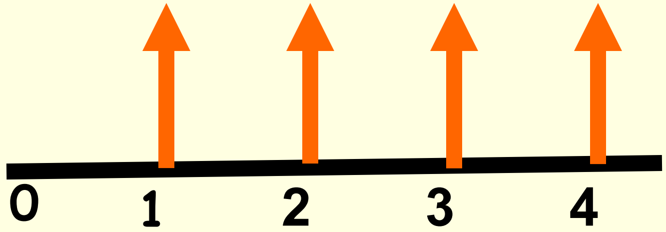
Conversion of linear gradient series to an annuity (uniform series)

Original Series



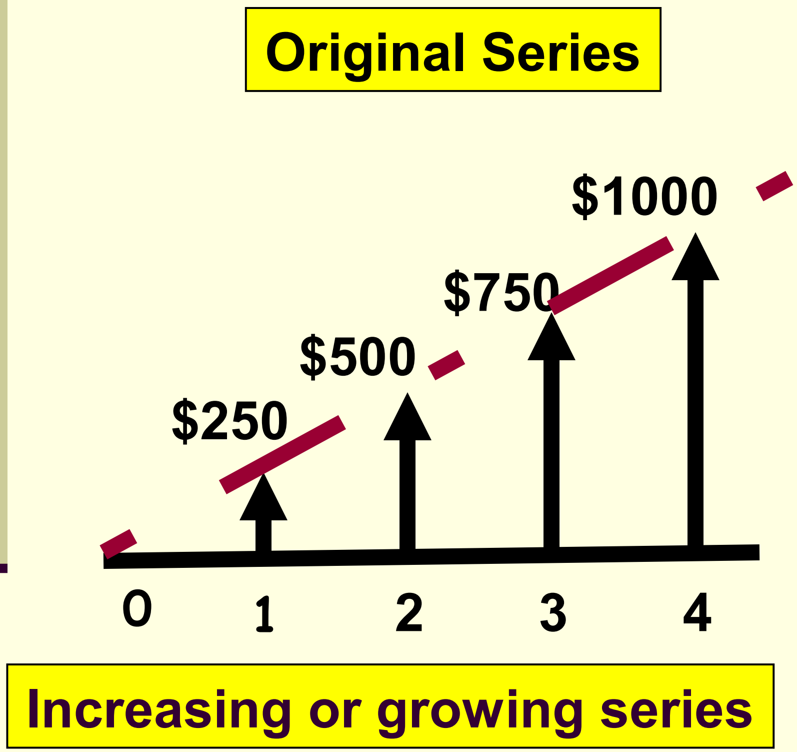
Increasing or growing series

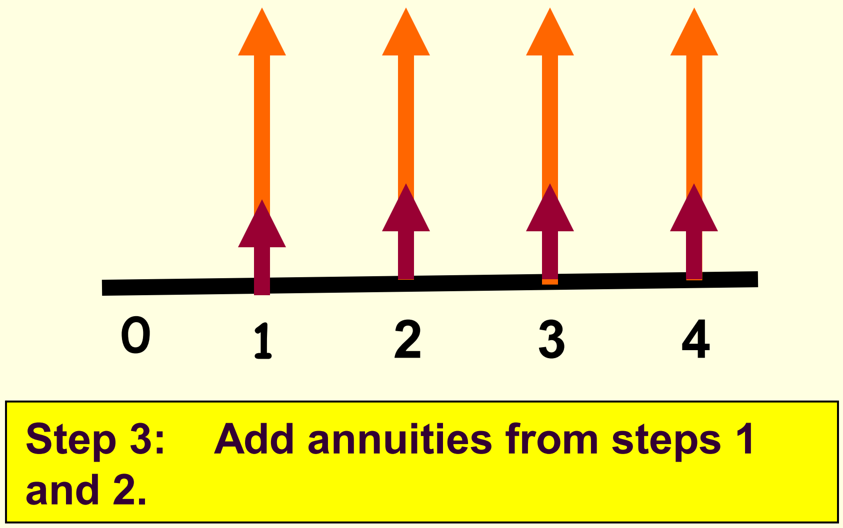




Step 2: Subtract first element from all elements including the first element and convert to an annuity.

Conversion of linear gradient series to an annuity (uniform series)





Arithmetic Gradient Series

<u>Composite series (two series in one): Increasing or growing series</u>

$P = 0 + G/(1+i)^2 + G/(1+i)^3 + G/(1+i)^4 \ldots + G/(1+i)^N$

OR $P = \text{sum } G(n-1)(1+i)^{N-1}$, $n = 1$ to N

- **Annual Equivalent** = A1 + G(A/G,i%,n)

  where G = gradient

  A1 = first element of series

  (A/G,i,N) = (1/i) -[N/[(1+i)N + 1]]
- **Present Worth** = A1(P/A1,i%,n) + G(P/G,i%,n)

  $(P/G,i,N) = \{[(1+i)^N - iN - 1] / [i^2(1+i)^N]\}$
- **Future Worth** = A1(F/A1,i%,n) + G(F/G,i%,n)

  $(F/G,i,N) = \{[(1+i)^N - 1]/i\} - N$

Arithmetic Gradient Series

Composite series (two series in one): Decreasing or decaying series

$P = 0 + G/(1+i)^2 + G/(1+i)^3 + G/(1+i)^4 + \ldots G/(1+i)^N$

OR $P = \text{sum } G(n-1)(1+i)^{N-1}$, $n = 1$ to N

- **Annual Equivalent** = A1 - G(A/G,i%,n)

  where G = gradient

  A1 = first element of series

  $(A/G,i,N) = (1/i) - [N/[(1+i)N - 1]]$

- **Present Worth =** A1(P/A1,i%,n) - G(P/G,i%,n)

  $(P/G,i,N) = \{[(1+i)^N - iN - 1]/ [i^2(1+i)^N]\}$

- **Future Worth =** A1(F/A1,i%,n) - G(F/G,i%,n)

  $(F/G,i,N) = \{[(1+i)^N – 1]/i\} - N$

Arithmetic Gradient Series

Title (name)	Formula (Need not be memorized)
Arithmetic gradient present worth factor: (P/G,i,N)	$[(1+i)^n\text{-}in\text{-}1]/ [i^2(1+i)^n]$
Arithmetic gradient uniform series factor: (A/G,i,N)	$[(1+i)^n\text{-}in\text{-}1]/[i(1+i)^n -1]$
Arithmetic gradient future worth factor: (F/G,i,N)	$\{[(1+i)^n - 1]/i\} – N$

Example: Increasing Arithmetic Gradient Series

Assume that your car is projected to have the following maintenance record during your 4-year studies at U of O. The rate of interest will be 10% compounded annually.

- Find the equivalent annual value of the maintenance charges which are:
  - \$300 (EOY1): \$350(EOY 2);\$400 (EOY 3) and \$450(EOY 4)

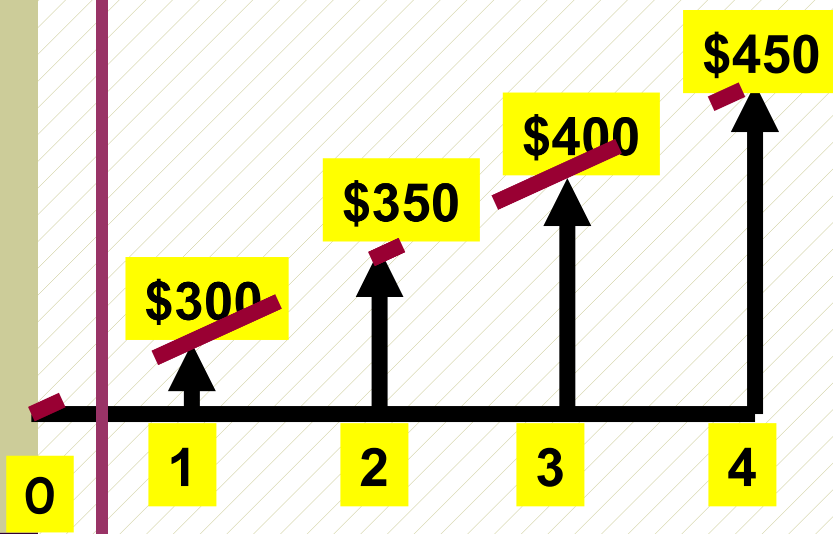
A = A1 + G(A/G,i%,N) = 300 + 50(A/G,10%,4)

= 300 + 50(1.381) = \$369.05

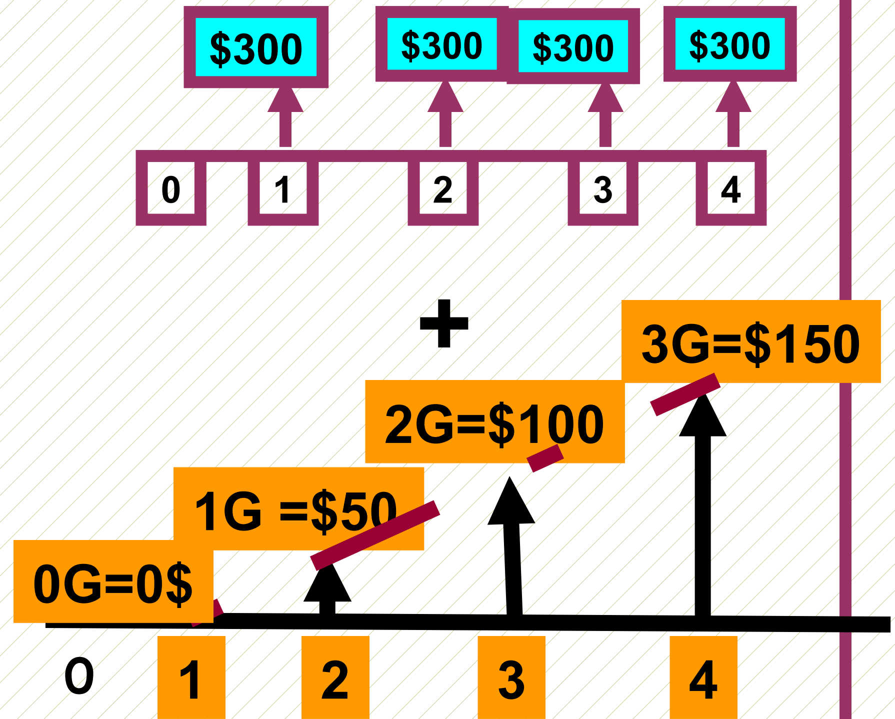
Note: EOY ≡ End of year

Annuity

- Actual Series



- Equivalent to:



DIS

n	(F/P,i%,n)	(P/					
1	1.1000	(					

Decreasing Arithmetic Gradient Series

- Annual Equivalent = A1 - G(A/G,i%,n)

  where G = gradient

  A1 = first element of series
- Present Worth (PW) = A1(P/A1,i%,n) - G(P/G,i%,n)
- Future Worth (FW) = A1(F/A1,i%,n) - G(F/G,i%,n)

Geometric Series

- Types of series
  - decaying
  - Exponential growth
- $(P/C,i,k,N) = \{C/(i-k)\}\{1-[(1+k)/(1+i)]^N\}$, if k ⑤ i
- $(P/C,i,k,N) = CN/(1+i) = CN/(1+k)$, $k = i$

- **‘C’ is the first term of the series which must be non-zero.**
- **‘i’ is the discount rate (MARR)**
- **‘k’ is the rate of growth or decay.**

Please refer to PowerPoint document #4 <u>Key Excel Functions in Engineering Economics</u> in the course website folder “ECO 1192 Documents on Virtual Campus”.

Geometric Series

Title (name)	Formulas
Geometric Series Present Worth Factor: (P/C,i,k,N) 1. If k% ≠ i% 2. If k% = i%	 1. $\{1 \div (i-k)\}\{1-[(1+k) \div (1+i)]^N\}$ 2. $N \div (1+i)$ OR $N \div (1+k)$
Geometric Series Future Worth Factor: (F/C,i,k,N) 1. If k% ≠ i% 2. If k% = i%	 1. $(1+i)^N$ (P/C,i,k,N) 2. $N(1+i)^{N-1}$ OR $N(1+k)^{N-1}$

Geometric Series: Example 1 ($i\% \neq k\%$)

If i (discount rate) = 10%; k (growth rate) = 15%; and N = 5 years, find

a) Series Present Worth

b) Series Future Worth

YEAR	CASH FLOW
0	0
1	100
2	$100(1+k)^1 = 100(1.15)^1 = 115$
3	$100(1+k)^2 = 100(1.15)^2 = 132.25$
4	$100(1+k)^3 = 100(1.15)^3 = 152.09$
5	$100(1+k)^4 = 100(1.15)^4 = 174.90$

Geometric Series:

Example: $i\% \neq k\%$; N=5 years and C=$100

<u>Present Worth</u>

Since k = 15% and i = 10% ($k\% \neq i\%$):

> ***Remember that k% is the rate of growth or decay of the cash flows and i% is the discount or compound rate***

PW = $C\{1 \div (i-k)\}\{1-[(1+k) \div (1+i)]^N\}$

$= 100\{1 \div (0.1-0.15)\}\{1-[(1+0.15) \div (1+0.1)]^5\}$

= **$498**

<u>Future Worth</u>

Since k = 15% and i =10%:

FW = $C\{1 \div (i-k)\}\{(1+i)^N - (1+k)^N\}$

$= 100\{1 \div (0.1- 0.15)\}\{(1+0.1)^5 - (1+0.15)^5\}$

= **$802** (which is also equal to 498(F/P,10%,5)

Geometric Series: Example 2

Example: If i%=k%=10%; N=5 years and C=$100

Present Worth

Since (k=10%) = (i=10%),

PW = CN/(1+i) **or** CN/(1+k)

where "C" ≡ 1st term (must be non-zero) of series

= 100(5)/(1+0.1)

= **$310.50**

Future Worth

Since k (=10%) = I (=10%),

FW = {CN/(1+k)}$(1+i)^N$

= 310.50(1.1^5)

= **$500**

Limiting Factors

INTEREST FACTOR	ALGEBRAIC EQUIVALENT	N ✡ ∞	i%=0
(F/P,i%,n)	$(1+i)^n$	Infinity	1
(P/F,i%,n)	$(1+i)^{-n}$	0	1
(F/A,i%,n)	$\{(1+i)^n - 1\}/i$	Infinity	N
(P/A,i%,n)	$\{(1+i)^n -1\}/\{i(1+i)^n\}$	1/i	N
(A/P,i%,n)	$\{i*(1+i)^n\}/\{(1+i)^n -1\}$	I	1/n
(A/F,i%,n)	$i/\{(1+i)^n – 1\}$	0	1/n

Special cases: n → ∞ or i% = 0

Example: Frequency Mismatch (Cash Flow and Interest Compounding)

10 end-of-year deposits of $1,000 are made to an account that pays 18% compounded quarterly.
What is the account balance after the 10^{th} deposit?

Method 1 : Find effective interest rate

- **From the effective rate formula:**
  $(1+r/m)^m - 1 = (1+0.18/4)^4 - 1 = 0.1925$
- **Therefore:** $F_{10} = 1{,}000(F/A, 19.25\%, 10) = \$25{,}015.36$

Method 2 : Find effective quarterly deposit

$A = 1000(A/F, 18\%/4, 4) = \$233.74$

$F_{40} = 233.74(F/A, 4.5\%, 40) = \$25{,}017.27$

Engineering Economics

ECO 1192

Topic 2: Time-value mechanics

Claude Théoret

University of Ottawa