

# Engineering Economics

ECO 1192

Topic 4: Rates of Return (IRR & ERR)

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# Recommended Readings

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- Fraser et al.\* chapter 3 and 5
  - Newnan et al. chapter 7
  - Park chapter 5 and 6

# Lecture objectives

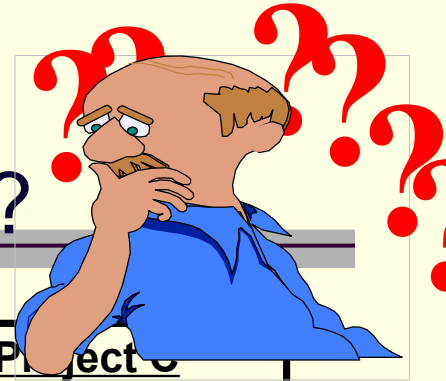
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- Present theory and applications of
  - Internal rate of return (IRR) method
  - External rate of return (ERR) method
- Compare IRR, ERR and MARR

# Student Numbers

- Please note that all grades
  - exams (final and mid-terms)
  - assignmentswill be recorded using your student number.
- My only reference will be the student number that you were provided by the registrar's office.
- Hence, it is very important that you indicate an **accurate student number** on all exams and assignments
  - Please DO NOT FORGET to darken the appropriate ovals on the exam/assignment answer sheets (Scantrons).
- No matching student number → 0% grade.

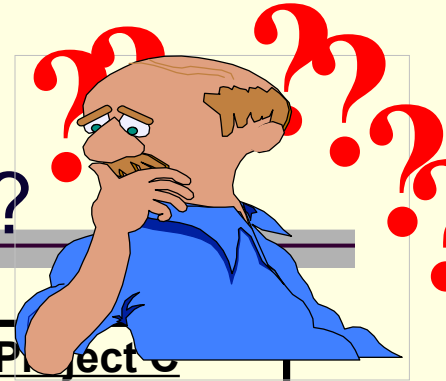
# Which projects are valid? Which project (if any) is best?



<u>Project parameters</u>	<u>Project A</u>	<u>Project B</u>	<u>Project C</u>
First cost (\$)	3,000	5,000	8,000
Annual cost (\$)	600	900	1300
Annual revenues (\$)	1500	1750	2000
Salvage value (\$)	0	-200	1000
Duration (years)	5	10	20
Interest rate (10%)	10	10	10

Last lecture: we used NPW, NFW and AEW to determine valid (acceptable) projects and the best project if A, B and C are mutually exclusive.

# Which projects are valid? Which project (if any) is best?



<u>Project parameters</u>	<u>Project A</u>	<u>Project B</u>	<u>Project C</u>
First cost (\$)	3,000	5,000	8,000
Annual cost (\$)	600	900	1300
Annual revenues (\$)	1500	1750	2000
Salvage value (\$)	0	-200	1000
Duration (years)	5	10	20
Interest rate (10%)	10	10	10

This lecture: we use the IRR and ERR methods to determine: valid (acceptable) projects and the best project (if A, B or C are mutually exclusive).

# Assumptions

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1. Discrete cash flow & compounding
2. Ownership or equity capital only (no debt capital)
3. No price changes
4. No uncertainty, no risk
5. No government (no taxes)
6. No imponderables or intangibles

# Examples: Investment Options

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## Investment A

- Zero coupon bond available for \$5,000 today.
  - No interest income is paid during the life of this investment.
- Bond will have a value of \$7,000 in exactly **two** years.

## Investment B

- \$5,000 purchase of Speedy Coulomb Co. common shares.
- Each common share currently sells for \$50 today and is expected to reach \$60 in **two** years.
- Each share pays its owner \$3 in annual dividends.

- **Which investment is better?**
- **Does the rate of return exceed MARR?**

# Investment Options

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- You could determine the better investment (A or B) by converting each alternative to a NPW, a NFW or a 2-year annuity.
- But the preferred approach is to
  - Calculate each investment's internal rate of return [IRRA=18%; IRRB=15%]
  - Compare individual rates of return and MARR
  - Select the “better” investment [A in this case]

# **Internal Rate of Return Method (IRR)**

# IRR Method: Characteristics and Assumptions

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1. Indirect calculation (i.e., trial and error)
2. Project cash inflows are assumed to be reinvested at the calculated rate of return (IRR)
  - not very practical in the real world (for example, where can money earn 10.55% today?)
3. **Possibility of multiple rates of return**
  - some positive, some negative (Descartes sign rule)
  - negative rates have no economic meaning
4. Based (i.e., dependent) on other summary methods
  - single sum (present and future) and the annual equivalent worth methods

# Possibility of multiple IRRs

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- Descartes' rule of signs
  - The total number of real-number roots is always less than or equal to the number of sign changes in the cash flow series.
- Norstroms' criterion
  - A single sign change in the series of cumulative cash flows which starts with a cash outflow (initial cost therefore negative) indicates that there is one positive root.

# Sign Rules and Multiple IRRs

<u>End of year</u>	<u>Descartes' Sign Rule</u> <u>Project's Net Cash Flow</u>	<u>Norstroms' Criterion</u> <u>Project's Cumulative Cash Flow</u>
0	-1,000	-1,000
1	1,100	+100
2	1,300	+1,400
3	-1,500	-100
4	1,400	+1,300
	<b>3 sign changes</b>	<b>3 sign changes</b>

# Calculating an investment's IRR ( $i^*$ )

Find  $i^*$ % such that

1. PW of cash inflows = PW of cash outflows
  - NPW = 0
2. FW of cash inflows = FW of cash outflows
  - NFW = 0
3. AEW of cash inflows = AEW of cash outflows
  - AEW = 0

Where

- NPW = Net Present Worth;
- NFW = Net Future Worth;
- AEW = Annual Equivalent Worth

# IRR Applications

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## Select

1. valid independent projects ( $i^*$ )
2. the better (or best) valid project between (among) competing alternatives using the **incremental** rate of return ( $i^{**}$ ).

# Example 1: A project's IRR

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1. You invest \$300 in a \$500 zero coupon bond (no interest income is paid during the ownership of the bond).
2. You receive \$500 after 10 years.
3. There is no inflation, risk or uncertainty, and no income taxes.

**Find the rate of return on your investment?**

# Example 1: A project's IRR

- You invest \$300 in a \$500 zero coupon bond (no interest income is paid during the ownership of the bond).
- You receive \$500 after 10 years.
- There is no inflation, risk or uncertainty, and no income taxes.
- **Find the rate of return on your investment?**

## Solution A: Use PW Method

- Find  $i^*$  such that  
PW of cash outflows = PW of cash inflows  
 $300 = 500(P/F, i^*, 10)$   
 $= 500(1+i^*)^{-10}$
- Solving for  $i^*$ :
- $i^* = 0.0524$  or 5.24%
- (Direct solution in this case)

# Example 1: A project's IRR

- You invest \$300 in a \$500 zero coupon bond (no interest income is paid during the ownership of the bond).
- You receive \$500 after 10 years.
- There is no inflation, risk or uncertainty, and no income taxes.
- **Find the rate of return on your investment?**

## Solution B: Use FW Method

- Find  $i^*$  such that
- FW of cash outflows = FW of cash inflows
- $300(1+i^*)^{10} = 500$
- Solving for  $i^*$ :
- $i^* = 0.0524$  or 5.24%
- (Direct solution in this case)

# Example 1: A project's IRR

- You invest \$300 in a \$500 zero coupon bond (no interest income is paid during the ownership of the bond).
- You receive \$500 after 10 years.
- There is no inflation, risk or uncertainty, and no income taxes.
- **Find the rate of return on your investment?**

## **Solution C: Use AEW Method**

Find  $i^*$  such that

AEW of outflows = AEW of inflows

$$300(A/P, i^*, 10) = 500(A/F, i^*, 10)$$

Solving for  $i^*$ :  $i^* = 0.0524$  or 5.24%

### **Note:**

The calculation of a project's internal rate of return is "method" independent:

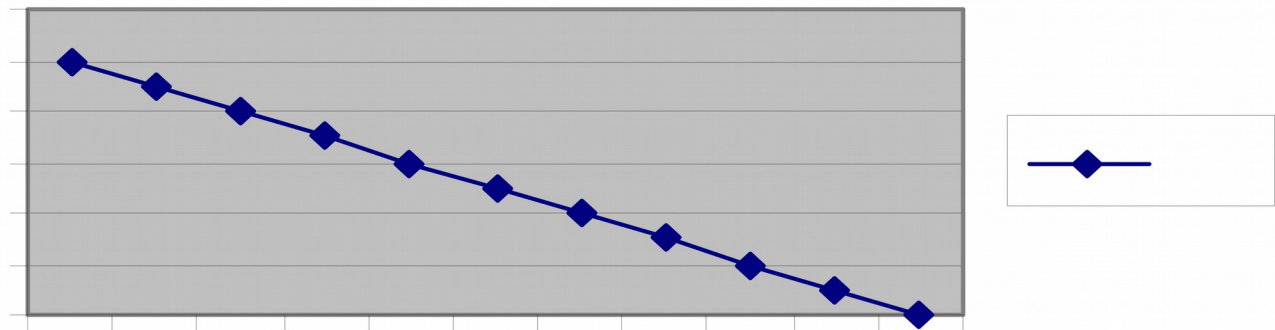
i.e., use NPW, NFW or AEW and you'll get the same answer.

# Example 1: Excel Solution

## Column J

		Cash	
<u>ROW</u>	<u>EOY</u>	<u>Flow</u>	
1	0	-300	
2	1	0	
3	2	0	
4	3	0	
5	4	0	
6	5	0	
7	6	0	
8	7	0	
9	8	0	
Please note that IRR is an Excel function,			
11	10	500	

# Example 1: A project's IRR



## Example 2: Finding a project's IRR Using the "Trial & Error" Method

<u>End-of-Year</u>	<u>Cash Flow</u>
0	-100
1	+20
2	+30
3	+20
4	+40
5	+40

## Example 2: Finding a project's IRR Using the "Trial & Error" Method

### **First attempt: Try $i\% = 10\%$**

- PW (Outflows) = 100
- PW (Inflows)  
=  $20(P/F, 10\%, 1) + 30(P/F, 10\%, 2) + 20(P/F, 10\%, 3)$   
+  $40(P/F, 10\%, 4) + 40(P/F, 10\%, 5)$   
= 110.16

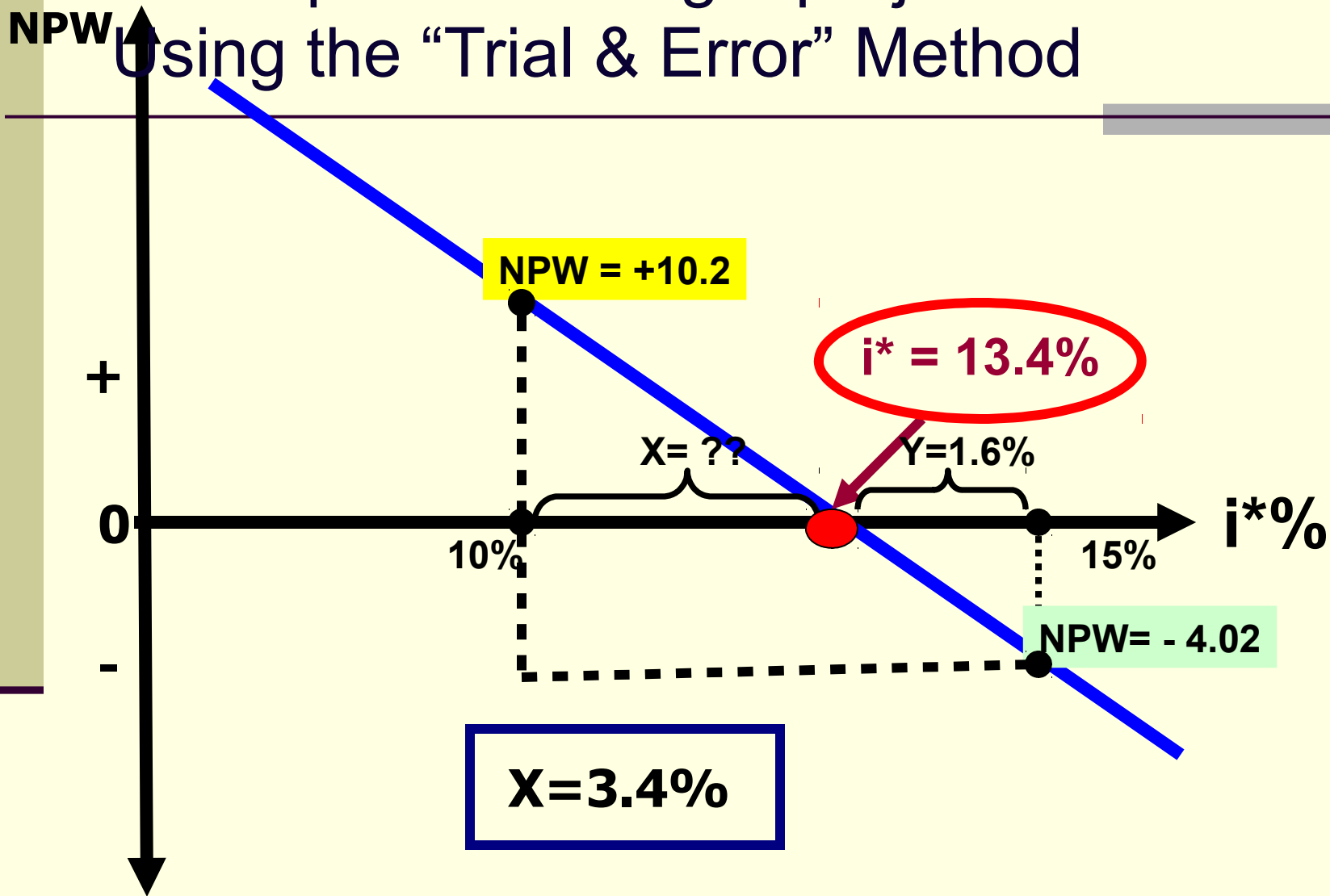
\*\*Incorrect interest rate; PW (Inflows) too low; need larger rate

### **Second attempt: Try $i^*% = 15\%$**

- PW (Outflows) = 100
- PW (Inflows)  
=  $20(P/F, 15\%, 1) + 30(P/F, 15\%, 2) + 20(P/F, 15\%, 3)$   
+  $40(P/F, 15\%, 4) + 40(P/F, 15\%, 5)$   
= -4.02 Incorrect interest rate; PW (Inflows) too high; try a rate between 10% and 15%
- By linear interpolation  $i^*% = 13.5\%$  (13.47%)

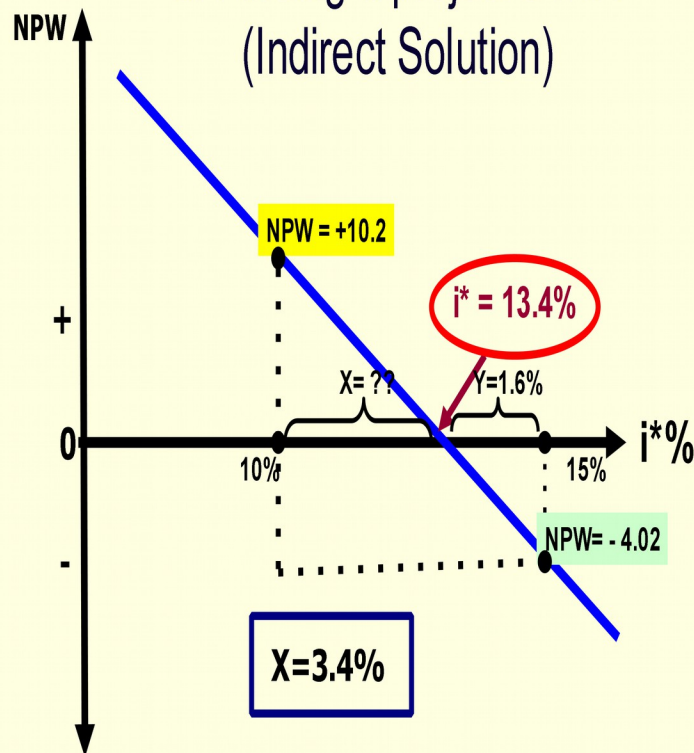
# Example 2: Finding a project's IRR

## Using the "Trial & Error" Method



# Example 2: Finding a project's IRR Using the "Trial & Error" Method

## 2. Finding a project's IRR (Indirect Solution)



from the congruency of triangles:

$$\begin{aligned} &\leq [x/(15\%-10\%)] \\ &= [10.2/(10.2+4.02)] \\ &\leq x = 10.2(15\%-10\%)/ \\ &\quad (10.2+4.02)] \\ &\leq x = 3.47\% \\ &\leq i^* = 10\% + x\% = 10\% \\ &\quad + 3.47\% = \underline{13.47\%} \end{aligned}$$

# Example 2: Excel Solution

<u>COLUMNS</u>			
<u>C</u>	<u>D</u>	<u>E</u>	
<u>ROWS</u>	<u>EOY</u>	<u>Cash Flow</u>	
<b>1</b>	<b>0</b>	<b>-100</b>	
<b>2</b>	<b>1</b>	<b>20</b>	
<b>3</b>	<b>2</b>	<b>30</b>	
<b>4</b>	<b>3</b>	<b>20</b>	
<b>5</b>	<b>4</b>	<b>40</b>	
<b>6</b>	<b>5</b>	<b>40</b>	
		<b>=IRR(E1:E6)</b>	<b>IRR=13.47%</b>

# Independent Projects: IRR Decision Rule

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If projects are independent, select all projects with individual internal rates of return ( $i^*$ %) at least equal to the MARR

- Assume no capital rationing or budgeting.

**Decision: If  $i^*_A > \text{MARR}$ ;  $i^*_B > \text{MARR}$  and  $i^*_C > \text{MARR}$ , select projects A, B and C.**

## Example 3: IRR & Independent Projects

<u>Cash Flows</u>		
<u>Year</u>	<u>Project A</u>	<u>Project B</u>
<b>0</b>	<b>-10</b>	<b>-20</b>
<b>1</b>	<b>15</b>	<b>28</b>

### Project A's IRR ( $i^*_A$ )

$$= (15 - 10)/10 = 5/10 = \mathbf{0.5 \text{ or } 50\%}$$

Project A is valid ( $i^*_A > \text{MARR}=6\%$ ).

### Project B's IRR ( $i^*_B$ )

$$= (28 - 20)/20 = 8/20 = \mathbf{0.4 \text{ or } 40\%}$$

Project B is valid ( $i^*_B > \text{MARR}=6\%$ ).

**Decision: Select both projects (if there is no capital budgeting)**

# Example 3: IRR & Two Mutually Exclusive Projects

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- Step 1
  - Ensure that at least one project has a valid individual rate of return ( $i^*$ ).
- Step 2
  - Determine the incremental rate of return ( $i^{**}$ ) by comparing the net cash flows of the two projects
    - Combine the individual cash flows of each project by subtracting one project's net cash flow from the other project's cash flow.

## Example 3: IRR & Two Mutually Exclusive Projects

### Decision Rule

1. If  $i^{**} < \text{MARR}$ , select the project with the smaller first cost (this project **MUST** have a valid individual rate of return).
2. If  $i^{**} > \text{MARR}$ , select the project with the larger first cost (this project **MUST** have a valid individual rate of return).
3. If  $i^{**} = \text{MARR}$ , select either project.

# WARNING !!!!

- If a single sum method is used (PW or FW) to determine the better project and the projects have different durations (lives), one or both projects MAY have to be repeated to obtain a COMMON period of analysis.
  - a project's adjusted first cost MUST be used to determine the better project.
- If the annual equivalent method is used to determine the better project, no adjustment is needed for projects with different durations or lives (irrelevant if  $N_A \neq N_B$ )
- It is highly recommended that AEW method be used to calculate the incremental IRR between two projects.

# Applications of the Incremental IRR ( $i^{**}$ )

Find  $i^{**}$  such that

1.  $PW(\Delta \text{ Cash Inflows}) = PW(\Delta \text{ Cash Outflows})$ ; or
2.  $FW(\Delta \text{ Cash Inflows}) = FW(\Delta \text{ Cash Outflows})$ ; or
3.  $AEW(\Delta \text{ Cash Inflows}) = AEW(\Delta \text{ Cash Outflows})$

Note:

1.  $\Delta \text{Cash Inflows} = \text{Project B's Cash Inflows} \text{ minus } \text{Project A's Cash Inflows}$  OR  $\text{Project A's Cash Inflows} \text{ minus } \text{Project B's Cash Inflows}$
2.  $\Delta \text{Cash Outflows} = \text{Project B's Cash Outflows} \text{ minus } \text{Project A's Cash Outflows}$  OR  $\text{Project A's Cash Inflows} \text{ minus } \text{Project B's Cash Inflows}$

# Example 3: IRR & Two Mutually Exclusive Projects

	<b>PW (<math>\Delta</math>Cash Inflows)</b> <b>= PW (<math>\Delta</math>Cash Outflows)</b> <del><math>28(P/F, i^{**}, 1) - 15(P/F, i^{**}, 1) = (20 - 10)</math></del> <del><math>13(P/F, i^{**}, 1) = 10</math></del>		
<b>Year</b>	<b>Project A</b>	<b>Project B</b>	
	<b>Solving: <math>i^* = 0.30</math> or 30% (direct solution)</b>		
<b>Decision</b>	-10	-20	
	<b>Since <math>i^{**}\%</math> (=30%) &gt; MARR,</b> <b>select the larger project <math>\rightarrow</math> B (<math>P_B = 20 &gt; P_A = 10</math>)</b>		
	1	15	28

# Validation of IRR Results

Use the PW Method to determine if Project A is better than Project B (assume that MARR=6%)

- PW (A) = \$10 gets \$15: Gain of \$5
- If the remainder (\$20 - \$10) is invested at MARR (6%) =  $10(0.06) = \$0.60$
- Future returns are  $(\$5 + \$0.60) = \mathbf{\$5.60}$
- For Project B, future returns are  $\$28 - \$20 = \mathbf{\$8.00}$
- Our objective is to maximize the total dollar returns for available funds
  - Project B (Gain = \$8) is better than the combination of project A and the remaining \$10 invested at 6% MARR.

# Pitfall: Incorrect Decision (1)

- ✂ Selecting the better project strictly on the basis of individual internal rates of return (select the project with the highest individual rate of return) **can** lead to an incorrect decision.
- ✂ In the previous example, you would have selected project A (i.e.,  $i^*=50\%$ ) over project B (i.e.,  $i^*=40\%$ )  
→ Incorrect decision

## Pitfall: Incorrect Decision (2)

- If the larger project (with the larger initial cost) has the larger individual IRR, then you can select the larger project as the better project.
- If  $(P_A > P_B) \ \& \ (i^*_A > i^*_B)$ , select project A based exclusively on individual rates of return
  - no incremental method is required in this case
- If  **$(P_A < P_B) \ \& \ (i^*_A > i^*_B)$  OR  $(P_A > P_B) \ \& \ (i^*_A < i^*_B)$** , you **MUST** use the incremental IRR method to determine the better project.

# Example 4: IRR & Two Mutually Exclusive Projects

<u>Project Paramete</u>	_____	_____

## Machine X

PW (Cash outflows) = PW (Cash inflows)

$$200 = 75 (P/A, i^*, 6) + 50 (P/F, i^*, 6)$$

Solving:  $i^* = 31.9\%$

## Machine Y

PW (Cash outflows) = PW (Cash inflows)

$$400 = 120(P/A, i^*, 12) + 150 (P/F, i^*, 12)$$

Solving:  $i^* = 29.1\%$ .

**Decision: Both machines X and Y are acceptable.**

# Example 4: IRR & Two Mutually Exclusive Projects

**Which project is better?  
X or Y?**

Find  $i^{**}\%$  such that the Annual equivalent ( $\Delta$ Cash Outflows) = Annual Equivalent ( $\Delta$  Cash Inflows)

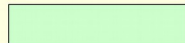
**OR**  $AEW_X = AEW_Y$

$$- 200(A/P, i^{**}, 6) + 75 + 50(A/F, i^{**}, 6) = - 400(A/P, i^{**}, 12) + 120 + 150 (A/F, i^{**}, 12)$$

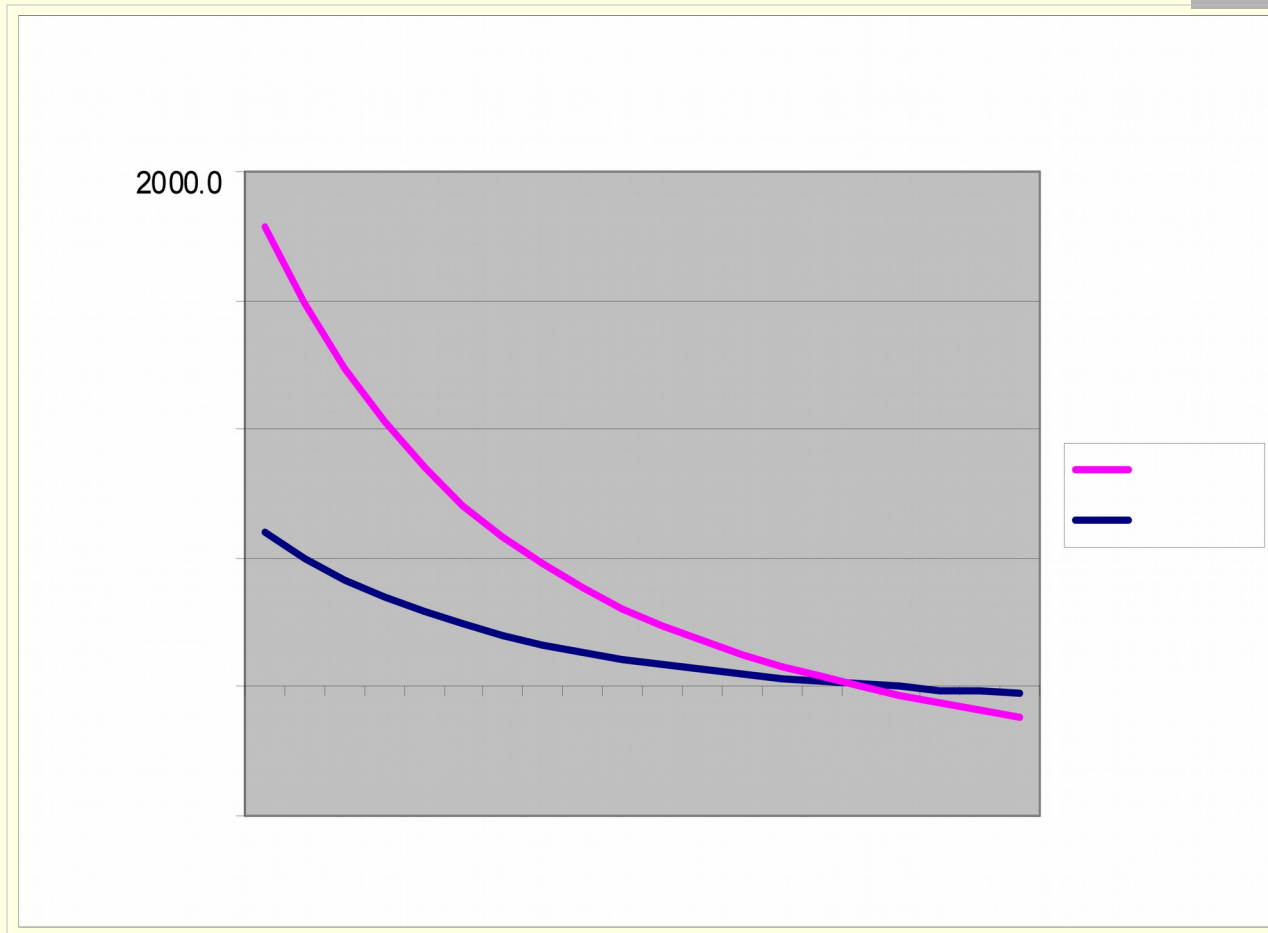
Solving:  $i^{**} = 29.1\%$

**Decision: Select Machine Y i.e., the larger project**

Project Parameters		



# Example 4: IRR & Two Mutually Exclusive Projects



# Example 4: IRR & Two Mutually Exclusive Projects

- The adjusted cost of Machine X is
  - $200 + 200(P/F, \text{MARR}, 6) = \$313$
- Machine X's effective (adjusted) cost **MUST** be compared to Machine Y's initial cost (which need not be adjusted because it was not repeated)
- In this case, Machine Y's initial cost (= \$400) is larger than the adjusted initial cost of Machine X (= \$313)

**Recommendation:** Always base  $i^{**}$  calculations on the AEW Method to avoid the issue of which project is “bigger”.

# Example 4: IRR & Two Mutually Exclusive Projects

- Machine Y's IRR = 29.1%
- The IRR for the following combination of investments is 20%
  - \$200 for Machine X now AND \$200 in 6 years
  - \$200 at MARR ( $200 @ 10\% = \$20$  annually)
- Machine Y's IRR (29.1%) exceeds the IRR (20%) for the combination of \$200 each in Machine X and at MARR over 12 years.

**Recommendation: Always base  $i^{**}$  calculations on the AEW Method to avoid the issue of “which project is bigger”.**

# Example 4: Excel's IRR Function & Two Mutually Exclusive Projects

		<u>Column C</u>	<u>Column D</u>	<u>Column E</u>
		Machine X Cash Flow	Machine Y Cash Flow	Differential Cash Flow
<u>ROWS</u>	<u>EOY</u>	<u>12 years</u>	<u>12 years</u>	<u>Machine Y - Machine X</u>
1	0	-200	-400	=D1-C1
2	1	75	120	=D2-C2
3	2	75	120	=D3-C3
4	3	75	120	=D4-C4
5	4	75	120	=D5-C5
6	5	75	120	=D6-C6
<b>7</b>	<b>6</b>	<b>=75+50-200</b>	120	=D7-C7
8	7	75	120	=D8-C8
9	8	75	120	=D9-C9
10	9	75	120	=D10-C10
11	10	75	120	=D11-C11
12	11	75	120	=D12-C12
<b>13</b>	<b>12</b>	<b>=75+50</b>	=120+150	=D13-C13
	IRR =	=IRR(C1:C13)	=IRR(D1:D13)	=IRR(E1:E13)
	IRR =	31.9%	29.1%	26.8%

# IRR & 3 or more mutually exclusive projects

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1. Calculate individual project IRRs until a valid ( $> 0\%$ ) rate is found.
2. Rank projects in ascending order of their initial or first costs (P).
3. Perform pair-wise comparisons starting with the two projects with the lowest first cost (ensuring that periods of analyses are respected) to obtain the incremental rate of return between these two projects; determine (based on the previous rule) which project is better; eliminate the losing project from contention

# IRR & 3 or more mutually exclusive projects

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4. Compare the better project from the previous comparison with the project with the next largest first cost; repeat step 3.
5. Continue pair-wise comparisons until all projects have been compared.
6. The best project is the 'winner' of the last pair-wise comparison.

# Example 5: IRR & 3 mutually exclusive projects

Parameters	Project A	Project B	Project C
First Cost (\$)	2000	4000	5000
Net Annual Revenue (\$)	410	639	700
Salvage Value (\$)	0	0	0
Life (years)	20	20	20
Interest Rate (%)	6%	6%	6%

Ensure that one project is valid.

Project A's IRR

$$2000 = 410(P/A, i^*, 20)$$

$$\text{Solve for } i^*: i^* = 20\%$$

Project A is valid since  $i^* (= 20\%) > \text{MARR}$

2. Compare Projects A & B

Find  $i^{**}$  (incremental IRR) such that

$$(4,000 - 2,000)$$

$$= (639 - 410)(P/A, i^{**}, 20)$$

$$\text{Solve } i^*: i^* = 9.6\%$$

$> \text{MARR} (= 6\%)$

Select Project B (the larger project) over Project A

# Example 5: IRR & 3 mutually exclusive projects

IRR & Mutually Exclusive Projects A, B, C

Parameters	Project A	Project B	Project C
First Cost (\$)	2000	4000	5000
Net Annual Revenue (\$)	410	639	700
Salvage Value (\$)	0	0	0
Life (years)	20	20	20
Interest Rate (%)	6%	6%	6%

# Summary

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1. If projects are mutually exclusive, the incremental IRR method ( $i^{**}$ ) is most likely required to determine the better/best project.
2. If projects are independent, only individual internal rates of return ( $i^*$ ) are required to determine acceptable projects.

# Example 6: IRR & 4 independent projects

- If projects are independent (without a budget constraint), all projects are valid based on their individual internal rates of return compared to the MARR=10%.
  - $i^*_A = 19\%$
  - $i^*_B = 15\%$
  - $i^*_C = 18\%$
  - $i^*_D = 16\%$

<u>Project</u>	<u>First Cost (\$)</u>	<u>Individual IRR (i*)</u>
<b>A</b>	<b>100</b>	<b>19%</b>
<b>B</b>	<b>175</b>	<b>15%</b>
<b>C</b>	<b>200</b>	<b>18%</b>
<b>D</b>	<b>250</b>	<b>16%</b>

# Example 6: IRR & 4 mutually exclusive projects

<u>Project</u>	<u>First Cost</u> (\$)	<u>Individual IRR</u> ( $i^*$ %)	<u>Incremental IRR</u> ( $i^{**}$ %)		
			<u>A</u>	<u>B</u>	<u>C</u>
A	100	19%	--	--	--
B	175	15%	9%	--	--
C	200	18%	17%	23%	--
D	250	16%	12%	17%	13%

# Example 7: IRR & 5 mutually exclusive projects

## IRR MATRIX: Example

PROJECT	INCREMENTAL RATES OF RETURN (%)				
	<u>A</u>	<u>B</u>	<u>C</u>	<u>D</u>	<u>E</u>
A	$i_A = 20$	--	--	--	--
B	40	$i_B^* = 30$	--	--	--
C	30	34	$i_C^* = 36$	--	--
D	34	19	14	$i_D^* = 26$	--
E	30	18	11	8	$i_E^* = 25$

July 22, 2005

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## IRR MATRIX (Previous Slide)

- IF MARR=40%, select STATUS QUO
- IF MARR=30%, select project C
- IF MARR=20%, select project C
- IF MARR=10%, select project D
- IF MARR=5%, select project E

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# External Rate of Return Method (ERR)

# External Rate of Return Method (ERR)

- Find the rate of return that provides the following equality:

$$\mathbf{FW \text{ (cash inflows)} = FW \text{ (cash outflows)}}$$


$$R_k(F/P, e\%, N-k) = D_k(P/F, e\%, k)(F/P, i^*, N)$$

where

- $R$  = annual receipts (cash inflows)
  - $D$  = annual disbursements (cash outflows)
  - $e\%$  = reinvestment rate
  - $k$  = end of year (0, 1, 2 ..... N)
  - $i^*\%$  = external rate of return
- The FW method **MUST** be used to calculate an ERR.

# ERR: Characteristics & Assumptions

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1. Direct method
  2. Single rate of return
  3. Based on the FUTURE WORTH method
  4. Net cash inflows are reinvested at the external (predetermined rate which is usually the MARR) rate of return
-  CAVEAT: Note that the Future Worth method is a single-sum method and an adjustment must be made to the period of analysis if projects have different lives

# ERR Applications

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1. Finding a project's external rate of return (known as a project's individual rate of return)
2. Selecting viable independent projects
3. Selecting the better (best) valid project between (among) competing alternatives; use the incremental rate of return method.

# Excel Functions: IRR and ERR Problems

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1. IRR and ERR problems can be solved using the Excel functions 'Goal Seek' or 'Solver'.
2. IRR problems can also be solved using the Excel function "IRR".
3. For ERR problems use the Excel function "MIRR".
  - Please note that the ERR calculated by the Excel function MIRR is incorrect a project has negative cash flows other than the initial investment.
4. Please check the course website for applications.

# ERR Decision Rule: Independent Projects

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## Decision Rule

- Select all projects with an individual
- $ERR \geq MARR$ 
  - remember that you have unlimited funds i.e., there is no capital rationing.

# Example 1: ERR & 1 project

<u>Year</u>	<u>Cash Flow</u>
<b>0</b>	<b>-100</b>
<b>1</b>	<b>+20</b>
<b>2</b>	<b>+30</b>
<b>3</b>	<b>+20</b>
<b>4</b>	<b>+40</b>
<b>5</b>	<b>+40</b>

- Assume that the reinvestment rate is 10% (= MARR)
- FW (Cash Outflows) =  $100(F/P, i^*, 5)$
- FW (Cash Inflows) =  $20(F/P, 10\%, 4) + 30(F/P, 10\%, 3)$   
 $+ 20(F/P, 10\%, 2) + 40(F/P, 10\%, 1) + 40(F/P, 10\%, 0)$   
 $= 177.41$ ; solve for  $i^*$ :  $i^* = 12.2\%$   
Since  $i^* = 12.2\% > \text{MARR} (=10\%)$ , this project is valid (acceptable).

# Example 1: Excel Function MIRR

		<b><u>COLUMN N</u></b>	
<b><u>ROWS</u></b>	<b><u>EOY</u></b>	<b><u>Project Cash Flows</u></b>	
<b>10</b>	<b>0</b>	<b>-100</b>	
<b>11</b>	<b>1</b>	<b>20</b>	
<b>12</b>	<b>2</b>	<b>30</b>	
<b>13</b>	<b>3</b>	<b>20</b>	
<b>14</b>	<b>4</b>	<b>40</b>	
<b>15</b>	<b>5</b>	<b>40</b>	
	<b>IRR =</b>	<b>=IRR(N10:N15)</b>	<b>= 13.47%</b>
	<b>ERR =</b>	<b>=MIRR(N10:N15,10%,10%)</b>	<b>= 12.1%</b>

# Calculating ERR using Excel's MIRR and IRR functions without negative cash flows

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- Excel's MIRR function calculates a project's ERR accurately if the project's only negative cash flow is its initial investment P.
- You can also calculate a project's ERR using Excel's IRR function using the following steps:
  - Convert all positive cash flows to future (end-of-project) values using the MARR.
  - Add all future (positive) cash flow values at the end of the project.
  - All positive (intermediate) cash flows transferred to the project's last year will now have \$0 value.
  - Use the IRR function to determine the project's ERR.
- Please refer to the following slide.

# Calculating ERR using Excel's MIRR and IRR functions without negative cash flows

## ERR METHOD WITHOUT NEGATIVE CASH FLOWS

MARR = 10%		SV = 1000		
N=10		Cash Flows	End-of-project	Cumulative
EOY	Cash Flows	Adjusted for SV	Cash Flow Values	Cash Flow Values
0	-20000	-20000	-20000.00	-20000.00
1	3000	3000	7073.84	0
2	3000	3000	6430.77	0
3	3000	3000	5846.15	0
4	3000	3000	5314.68	0
5	3000	3000	4831.53	0
6	3000	3000	4392.30	0
7	3000	3000	3993.00	0
8	3000	3000	3630.00	0
9	3000	3000	3300.00	0
10	<u>3000</u>	<u>4000</u>	4000.00	48812.27
ERR =		9.33%		9.33%
IRR =		8.64%		

# Calculating ERR using Excel's MIRR and IRR functions with negative cash flows

- Excel's MIRR function calculates a project's ERR inaccurately when there are negative flows other than the initial investment.
- You can also calculate a project's ERR using Excel's IRR function using the following steps:
  - Convert all negative cash flows throughout the project to their equivalent present values using MARR and add to the initial cost (P) of the project.
  - Convert all positive cash flows to future (end-of-project) values using MARR.
  - Add all future (positive) cash flow values at the end of the project.
  - All positive (intermediate) cash flows transferred to the project's last year will now have \$0 value.
  - Use the IRR function to determine the project's ERR.
- Please refer to the following slide.



# ERR Application: Two Mutually Exclusive Projects

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Two steps to this process:

1. Ensure that one project has a valid individual external rate of return (not necessary to calculate the individual for competing alternatives because the valid project will surface as the best project if other projects have ERRs  $<$  MARR).
2. Determine the incremental ERR for differences in project cash outflows and cash inflows.

# ERR Application: Two Mutually Exclusive Projects

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## Decision Rule

- If the incremental ERR exceeds the MARR, select the project with the larger first cost

**WARNING:** Since the ERR method is based on FW, pair-wise comparisons require a common period of analysis.

# Example 2: ERR & 2 independent projects

<u>Parameters</u>	<u>Machine X</u>	<u>Machine Y</u>
First Cost (\$)	200	400
Net annual cash inflow (\$)	75	120
Salvage Value (\$)	50	150
Life (years)	6	12

Machine X

~~MARK~~ - **10%**

## Example 2: ERR & 2 independent projects

---

- Individual ERRs ( $i^*$ )
  - Machine X = 16.5%
  - Machine Y = 17.3%

### **Decision (No capital rationing)**

Because both projects have individual ERR's > MARR=10%, select both projects.

# ERR Application: Two Mutually Exclusive Projects

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- ✂ If the larger project (larger initial cost) has the higher external rate of return, then it would automatically be chosen as the better project
- ✂ Please note that the definition of initial cost  $\neq$  “P” when projects have different durations
- . Because the larger project does NOT have the larger initial cost, we must calculate the incremental external rate of return.

# Example 2: ERR & 2 mutually exclusive projects

ERR =  $\frac{\text{FW}(\Delta\text{Cash inflows}) - \text{FW}(\Delta\text{Cash outflows})}{\text{FW}(\Delta\text{Cash inflows})}$

<u>Parameters</u>	<u>Machine X</u>	<u>Machine Y</u>
First Cost (\$)	200	400
Net annual cash inflow (\$)	75	120
Salvage Value (\$)	50	150
Life (years)	6	12

# Example 2: Excel MIRR Function and 2 mutually exclusive projects

## Column J

ROW 27

	<u>Column I</u>	<u>Machine Y</u>	<u>Column K</u>
	<u>Machine X Cash Flow</u>	<u>Cash</u>	<u>Differential Cash Flow</u>
<u>EOY</u>	<u>12 years</u>	<u>Flow</u>	<u>Machine Y - Machine X</u>
0	-200	-400	=J27-I27
1	75	120	=J28-I28
2	75	120	=J29-I29
3	75	120	=J30-I30
4	75	120	=J31-I31
5	75	120	=J32-I32
6	=75+50-200	120	=J33-I33
7	75	120	=J34-I34
8	75	120	=J35-I35
9	75	120	=J36-I36
10	75	120	=J37-I37
11	75	120	=J38-I38
12	=75+50	=120+150	=J39-I39
IRR =	=IRR(I27:I39)	=IRR(J27:J39)	=IRR(K27:K39)
ERR =	=MIRR(I27:I39,10%,10%)	=MIRR(J27:J39,10%,10%)	=MIRR(K27:K39,10%,10%)

ROW 39

# Example 2: Excel MIRR Function and 2 mutually exclusive projects

	<u>Column I</u>	Machine Y	<u>Column K</u>
	Machine X Cash Flow	Cash	Differential Cash Flow
<u>EOY</u>	<u>12 years</u>	<u>Flow</u>	<u>Machine Y - Machine X</u>
<b>0</b>	<b>-200</b>	<b>-400</b>	<b>-200</b>
<b>1</b>	<b>75</b>	<b>120</b>	<b>45</b>
<b>2</b>	<b>75</b>	<b>120</b>	<b>45</b>
<b>3</b>	<b>75</b>	<b>120</b>	<b>45</b>
<b>4</b>	<b>75</b>	<b>120</b>	<b>45</b>
<b>5</b>	<b>75</b>	<b>120</b>	<b>45</b>
<b>6</b>	<b>-75</b>	<b>120</b>	<b>195</b>
<b>7</b>	<b>75</b>	<b>120</b>	<b>45</b>
<b>8</b>	<b>75</b>	<b>120</b>	<b>45</b>
<b>9</b>	<b>75</b>	<b>120</b>	<b>45</b>
<b>10</b>	<b>75</b>	<b>120</b>	<b>45</b>
<b>11</b>	<b>75</b>	<b>120</b>	<b>45</b>
<b>12</b>	<b>125</b>	<b>270</b>	<b>145</b>
<b>IRR =</b>	<b>31.9%</b>	<b>29.1%</b>	<b>26.8%</b>
<b>ERR =</b>	<b>16.5%</b>	<b>17.3%</b>	<b>17.1%</b>

## Example 3: ERR & 3 independent projects

<u>Parameters</u>	<u>Project A</u>	<u>Project B</u>	<u>Project C</u>
<b>First Cost (\$)</b>	<b>2000</b>	<b>4000</b>	<b>5000</b>
<b>Net Annual Revenues (\$)</b>	<b>410</b>	<b>639</b>	<b>700</b>
<b>Salvage Value (\$)</b>	<b>0</b>	<b>0</b>	<b>0</b>
<b>Life (years)</b>	<b>20</b>	<b>20</b>	<b>20</b>
<b>MARR (%)</b>	<b>6</b>	<b>6</b>	<b>6</b>

# Example 3: ERR & 3 independent projects

1. Ensure that one project is valid (its ERR > MARR).  
Project A:  $2000(F/P, i^*, 20) = 410(F/A, 6\%, 20)$   
 $i_A^* = 10.6\%$ ; Project A is valid
2. Compare Projects, A & B; find  $i^*$  such that  
 $(4000-2000)(F/P, i^*, 20) = (639-410)(F/A, 6\%, 20)$   
 $i^{**} = 7.45\%$ ; select Project B
3. Compare projects B & C; find  $i^*$  such that  
 $(5000-4000)(F/P, i^*, 20) = (700-639)(F/A, i^*, 20)$   
 $i^{**} = 4.12\%$ ; select the smaller project (i.e., B)

# Example 3: ERR & 3 mutually exclusive projects

Project	First Cost (\$)	Individual and Incremental ERRs (%)		
A	2,000	$i^*_A = 10.6\%$	--	--
B	4,000	7.5	$i^*_B = 9.3$	--
C	5,000	6.5	4.1	$i^*_C = 8.5$

**Decision: Project B is the best project if A, B and C are mutually exclusive (MARR = 6%).**

# IRR, ERR and MARR Comparisons

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1. ERR is always between IRR and MARR.
2. IF  $IRR < MARR$ , then  $IRR < ERR < MARR$
3. IF  $IRR > MARR$ , then  $IRR > ERR > MARR$
4. IF  $IRR = MARR$ , then  $IRR = ERR = MARR$ .

# Summary: Excel Functions

		<u>Column E</u>		
	<u>EOY</u>	<u>Net Cash Flow</u>		
Row 3	0	-300000		
4	1	60000		
5	2	60000		
6	3	60000		
7	4	60000		
8	5	60000		
9	6	60000		
10	7	60000		
11	8	60000		
12	9	60000		
13	10	=60000+50000		
	NPV =	=NPV(10%,E4:E13)+E3	=PV(10%,10,-60000,-50000)+E3	87951.19081375 77
	NFW =	=E16*(1.1)^10	=FV(10%,10,-60000,300000)+50000	228122.73803
	A =	=-PMT(10%,10,E16)	=-PMT(10%,10,-300000,50000)+60000	14313.65127937 21
	IRR =	=IRR(E3:E13)		0.161131859770 504
	MIRR =	=MIRR(E3:E13,10%,10%)		0.128647356225 23

# Summary: Excel Functions

		<u>Column E</u>
	<u>EOY</u>	<u>Net Cash Flow</u>
<u>Row 3</u>	<b>0</b>	<b>-300000</b>
<b>4</b>	<b>1</b>	<b>60000</b>
<b>5</b>	<b>2</b>	<b>60000</b>
<b>6</b>	<b>3</b>	<b>60000</b>
<b>7</b>	<b>4</b>	<b>60000</b>
<b>8</b>	<b>5</b>	<b>60000</b>
<b>9</b>	<b>6</b>	<b>60000</b>
<b>10</b>	<b>7</b>	<b>60000</b>
<b>11</b>	<b>8</b>	<b>60000</b>
<b>12</b>	<b>9</b>	<b>60000</b>
<b>13</b>	<b>10</b>	<b>110000</b>
	<b>NPV =</b>	<b>\$87,951</b>
	<b>NFW =</b>	<b>\$228,123</b>
	<b>A =</b>	<b>\$14,314</b>
	<b>IRR =</b>	<b>16.1%</b>
	<b>MIRR =</b>	<b>12.9%</b>

# Summary Problem

Parameters			
First Cost (\$)			

# Summary Measures: Decisions

<u>Summary Measures</u>	<u>Independent Projects</u>	<u>Mutually Exclusive Projects</u>
Present Worth	A, B and C	C
Future Worth	A, B and C	C
Annual Equivalent	A, B and C	C
IRR	??	??
ERR	??	??
Simple Payback		
Discounted Payback		

# Individual IRRs (PW-based)

## Project A

Find  $i^*$ % such that

$$PW(A)^{\text{cash inflows}} = PW(A)^{\text{cash outflows}}$$

$$2000(P/A, i^*, 10) = 10000$$

Solve for  $i^*$ :  $i^* = 15.1\%$  (A is acceptable)

## Project B

Find  $i^*$ % such that

$$PW(A)^{\text{cash inflows}} = PW(A)^{\text{cash outflows}}$$

$$3850(P/A, i^*, 10) = 18000 + 1000(P/F, i^*, 10)$$

Solve for  $i^*$ :  $i^* = 16.6\%$  (B is acceptable)

# Individual IRRs (PW-based)

## Project C

Find  $i^*$ % such that

$$PW(A)^{\text{Cash inflows}} = PW(A)^{\text{Cash outflows}}$$

$$2800(P/A, i^*, 20) + 400(P/G, i^*, 20) + 3000(P/F, i^*, 20) = 30000$$

Solve for  $i^*$ :  $i^* = 15.4\%$  (C is acceptable)

**Decision: Projects A, B and C are acceptable as their individual IRRs > MARR(=10%)**

# IRR Method & Mutually Exclusive Projects A, B and C

## Compare the two smallest projects, A and B

- Find  $i^*$  such that the  $PW(A) = PW(B)$
- $2,000(P/A, i^*, 10) + 0 - 10,000$   
 $= 3,850(P/A, i^*, 10) - 1,000(P/F, i^*, 10) - 18,000$
- Solve for the incremental internal rate  $i^*$ ;  $i^* = 0.184$
- Since the incremental IRR exceeds MARR (18.4% versus 10%), select the larger project i.e., B.
- This step was not required as Project A's (the smaller project) internal rate of return (=15.1%) was smaller than Project B's (the larger project) internal rate (16.6%)
- You could have concluded that the larger project (with the higher rate of return) was the better project.

# IRR Method & Mutually Exclusive Projects A, B and C

- Compare B and C (over a 20-year period)
- Find  $i^*$  such that the  $PW(B) = PW(C)$   
$$3,850(P/A, i^*, 20) - 1,000[(P/F, i^*, 10) + (P/F, i^*, 20)]$$
$$- 18,000[1 + (P/F, i^*, 10)]$$
$$= 2,800(P/A, i^*, 20) + 400(P/G, i^*, 20)$$
$$+ 3,000(P/F, i^*, 20) - 30,000$$
- Solve for  $i^*$ :  $i^* = 0.147$  or 14.7%  
Decision: Since the incremental IRR exceeds the MARR (14.7% versus 10%), select the larger project i.e., C

# Individual ERRs (FW-based)

## **Project A**

Find  $i^*$ % such that

$$FW(A)^{\text{cash inflows}} = FW(A)^{\text{cash outflows}}$$

$$2000(F/A, 10\%, 10) = 10000(F/P, i^*, 10)$$

Solve for  $i^*$ :  $i^* = 0.123$  or 12.3% (Acceptable)

## **Project B**

Find  $i^*$ % such that

$$FW(B)^{\text{cash inflows}} = FW(B)^{\text{cash outflows}}$$

$$3850(F/A, 10\%, 10) - 1000 = 18000(F/P, i^*, 10)$$

Solve for  $i^*$ :  $i^* = 0.129$  or 12.9% (Acceptable)

# Individual ERRs (FW-based)

## Project C

Find  $i^*$ % such that

$$FW(A)^{\text{cash inflows}} = FW(A)^{\text{cash outflows}}$$

$$2800(F/A, 10\%, 20) + 400(F/G, 10\%, 20) + 3000 = 30000(F/P, i^*\%, 20)$$

Solve for  $i^*$ :  $i^* = 0.124$  or 12.4% (Acceptable)

**Decision: Projects A, B and C are acceptable as their individual ERRs > MARR(=10%)**

# Incremental ERR and Projects A & B

- Find  $i^*$  such that  $FW(A) = FW(B)$   
 $2,000(F/A, 10\%, 10) + 0 - 10,000(F/P, i^*\%, 10)$   
 $= 3,850(F/A, 10\%, 10) - 1,000 - 18,000 (F/P, i^*\%, 10)$
- Solve for the incremental ERR;  $i^* = 0.135$  or 13.5%
- Since the incremental ERR exceeds MARR (13.5% versus 10%), select Project B i.e., the larger project between A and B.
- Note that this step was not required as Project A's (the smaller project) ERR was smaller than Project B's (the larger project)
- Decision: Project B (the larger project with the higher external rate) is the better project.

## Incremental ERR and Projects B & C: 20-year period of analysis

- Find  $i^*$  such that the  $FW(B) = FW(C)$
- $3850(F/A, 10\%, 20) - 1,000[1+(F/P, 10\%, 10)]$   
 $- 18,000[1+(P/F, 10\%, 10)](F/P, i^*, 20)$   
 $= 2,800(F/A, 10\%, 20)+400(F/G, 10\%, 20)$   
 $+ 3,000 - 30,000(F/P, i^*, 20)$
- Solve for the incremental external rate  $i^*$ ;  
 $i^* = 0.158$  or 15.8%
- Since the incremental ERR exceeds MARR (15.8% versus 10%), select C i.e., the larger of C and B

# Summary Measures: Decisions

<u>Summary Measures</u>	<u>Independent Projects</u>	<u>Mutually Exclusive Projects</u>
<b>Present Worth</b>	<b>A, B and C</b>	<b>C</b>
<b>Future Worth</b>	<b>A, B and C</b>	<b>C</b>
<b>Annual Equivalent</b>	<b>A, B and C</b>	<b>C</b>
<b>IRR</b>	<b>A, B and C</b>	<b>C</b>
<b>ERR</b>	<b>A, B and C</b>	<b>C</b>
<b>Simple Payback</b>	<b>tbd</b>	<b>tbd</b>
<b>Discounted Payback</b>	<b>tbd</b> Topic 4: IRR & ERR	<b>tbd</b>

# Engineering Economics

ECO 1192

Topic 4: Rates of Return (IRR & ERR)

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