All solutions must include a carefully written explanation.

1. [10 MARKS.] Let A and B be subsets of a vector space V.

(a) Prove that
$$\operatorname{span}(A \cap B) \subseteq \operatorname{span}(A) \cap \operatorname{span}(B).$$

(b) Give an example of sets A and B and a vector space V such that $\operatorname{span}(A \cup B)$ is *not* a subset of $\operatorname{span}(A) \cup \operatorname{span}(B)$.

2. [10 MARKS.] Let
$$W = \{f(x) \in P_2(\mathbb{R}) : f(5) = 0\}.$$

(a) Prove that W is a subspace of $P_2(\mathbb{R})$.

(b) Find a basis for W.

3. [10 MARKS.] Let $T : P_2(\mathbb{R}) \to \mathbb{R}$ be the linear transformation defined by
$$T(f(x)) = \int_0^1 f(t)\,\mathrm{d}t.$$

(a) Find the rank and nullity of T.

(b) Find a basis of the nullspace of T.

4. [10 MARKS.] Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation such that $T(1,1) = (2,3)$ and $T(1,-1) = (-1,2)$.

(a) Find the matrix representation $[T]_\alpha$ of T with respect to the standard ordered basis
$$\alpha = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$
of $\mathbb{R}^2$.

(b) Compute $T(-3,2)$.

5. [10 MARKS.] Let $\{u, v, w\}$ be a linearly independent set of vectors in a vector space V over $\mathbb{R}$. Prove that

$$\{u + v, v + w, w + u\}$$

is also linearly independent.