All solutions must include a carefully written explanation.

1. [5 MARKS.] Determine whether the following subsets of $\mathbb{R}^3$ are subspaces of $\mathbb{R}^3$.

(a) Is
$$W_1 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 = 3a_2 \quad \text{and} \quad a_3 = -a_2\}$$
a subspace of $\mathbb{R}^3$?

(b) Is
$$W_2 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : 5a_1^2 - 3a_2^2 + 6a_3^2 = 0\}$$
a subspace of $\mathbb{R}^3$?

2. [5 MARKS.] Determine whether the given set is a basis for the given vector space.

(a) Is $\{(1, -1, 2), (2, 0, 1), (-1, 2, -1)\}$ a basis of $\mathbb{R}^3$?

(b) Is $\left\{ \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ -4 & 4 \end{bmatrix} \right\}$ a basis of $M_{2\times 2}(\mathbb{R})$?

3. [5 MARKS.] Find the basis of the subspace of $\mathbb{R}^4$ which is a subset of the solution set of
$$x_1 - x_2 + x_3 - x_4 = 0$$
$$2x_1 - 3x_2 + x_3 + x_4 = 0.$$

4. [5 MARKS.] Let $V = M_{2\times 2}(\mathbb{R})$,
$$W_1 = \left\{ \begin{bmatrix} a & a \\ b & c \end{bmatrix} : a, b, c \in \mathbb{R} \right\}, \quad \text{and} \quad W_2 = \left\{ \begin{bmatrix} 0 & 0 \\ d & e \end{bmatrix} : d, e \in \mathbb{R} \right\}.$$
Find the dimensions of W_1, W_2, $W_1 + W_2$, and $W_1 \cap W_2$.

5. [5 MARKS.] Let $T : V \to W$ be a linear transformation. Let $\{w_1, w_2, \ldots, w_k\}$ be a linearly independent subset of $R(T)$. Prove that if $S = \{v_1, v_2, \ldots, v_k\}$ is chosen so that $T(v_i) = w_i$ for each i, $1 \leq i \leq k$, then S is linearly independent.

6. [5 MARKS.] Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by $T(a_1, a_2) = (a_1 - a_2, a_1, 2a_1 + a_2)$. Let $\alpha = \{(1, 2), (2, 3)\}$ and $\beta = \{(1, 1, 0), (0, 1, 1), (2, 2, 3)\}$ be bases of $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively. Find $[T]_\alpha^\beta$.